

# DECOMPOSITIONS, APPROXIMATE STRUCTURE, TRANSFERENCE, AND THE HAHN-BANACH THEOREM

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ABSTRACT. We discuss three major classes of theorems in additive and extremal combinatorics: decomposition theorems, approximate structure theorems, and transference principles. We also show how the finite-dimensional Hahn-Banach theorem can be used to give short and transparent proofs of many results of these kinds. Amongst the applications of this method is a much shorter proof of one of the major steps in the proof of Green and Tao that the primes contain arbitrarily long arithmetic progressions. In order to explain the role of this step, we include a brief description of the rest of their argument. A similar proof has been discovered independently by Reingold, Trevisan, Tulsiani and Vadhan [RTTV].

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## 1. INTRODUCTION

This paper has several purposes. One is to provide a survey of some of the major recent developments in the rapidly growing field that has come to be known as additive combinatorics, focusing on three classes of theorems: decomposition theorems, approximate structural theorems and transference principles. (An explanation of these phrases will be given in just a moment.) A second is to show how the Hahn-Banach theorem leads to a simple and flexible method for proving results of these three kinds. A third is to demonstrate this by actually giving simpler proofs of several important results, or parts of results. One of the proofs we shall simplify is the proof of Green and Tao that the primes contain arbitrarily long arithmetic progressions [GT1], which leads to the fourth purpose of this paper: to provide a partial guide to their paper. We shall give a simple proof of a result that is implicit in their paper, and made explicit in a later paper of Tao and Ziegler, and then we shall explain informally how they use this result to prove their famous theorem. A proof along similar lines has been discovered independently by Reingold, Trevisan, Tulsiani and Vadhan [RTTV]. We have tried to design this paper so that the reader who is just interested in the Green-Tao theorem can get away with reading only a small part of it. However, the earlier sections of the paper provide considerable motivation for the later arguments, so such a reader would be well-advised at least to skim the sections that are not strictly speaking necessary.

Now let us describe the classes of theorems that will principally concern us. By a *decomposition theorem* we mean a statement that tells us that a function  $f$  with certain properties can be decomposed as a sum  $\sum_{i=1}^k g_i$ , where the functions  $g_i$  have certain other properties. There are two kinds of decomposition theorem that have been particularly useful. One kind says that  $f$  can be written as  $\sum_{i=1}^k g_i + h$ , where the functions  $g_i$  have some explicit description and  $h$ , the “error term” is in a useful sense small.

Another kind, which is closely related, brings us to our second class of results. An *approximate structure theorem* is a result that says that, under appropriate conditions, we can write a function  $f$  as  $f_1 + f_2$ , where  $f_1$  is “structured” in some sense, and  $f_2$  is “quasirandom”. The rough idea is that the structure of  $f_1$  is strong enough for us to be able to analyse it reasonably explicitly, and the quasirandomness of  $f_2$  is strong enough for many properties of  $f_1$  to be unaffected if we “perturb” it to  $f_1 + f_2$ . Often, in order to obtain stronger statements about the structure and the quasirandomness, one allows also a small  $L_2$ -error: that is, one writes  $f$  as  $f_1 + f_2 + f_3$  with  $f_1$  structured,  $f_2$  quasirandom, and  $f_3$  small in  $L_2$ .

A *transference principle* is a statement to the effect that a function  $f$  that belongs to some space  $X$  of functions can be approximated by a function  $g$  that belongs to another space  $Y$ . Such a statement is useful if the functions in  $Y$  are easier to handle than the functions in  $X$  and the approximation is of a kind that preserves the properties that one is interested in. As we shall see later, a transference principle is the fundamental step in the proof of Green and Tao. For now, let us merely note that a transference principle is a particular kind of decomposition theorem: it tells us that  $f$  can be written as  $g + h$ , where  $g \in Y$  and  $h$  is small in an appropriate sense.

As well as the Green-Tao theorem, we shall discuss several other results in additive combinatorics. One is a structure theorem proved by Tao in an important paper [T1] that gives a discretization of Furstenberg’s ergodic-theory proof [Fu] of Szemerédi’s theorem [S1], or more precisely a somewhat different ergodic-theory argument due to Host and Kra [HK05]. We shall give an alternative proof of (a slight generalization of) this theorem, and give some idea of how it can be used to prove other results. Amongst these other results are Roth’s theorem [Rot], which states that every set of integers of positive upper density contains an arithmetic progression of length 3, and Szemerédi’s regularity lemma [S2], a cornerstone of extremal graph theory, which shows that every graph can be approximated by a disjoint union of boundedly many quasirandom graphs (and which is a very good example of an approximate structure theorem).

The remaining sections of this paper are organized as follows. The next section introduces several norms that are used to define quasirandomness. Strictly speaking, it is independent of much of the rest of the paper, since many of our results will be rather general ones about norms that satisfy various hypotheses. However, for the reader who is unfamiliar with the basic concepts of additive combinatorics it may not be obvious that these

hypotheses are satisfied except in one or two very special cases: section 2 should convince such a reader that the general results can be applied in many interesting contexts.

In section 3, we introduce our main tool, the finite-dimensional Hahn-Banach theorem, and we give one or two very easy consequences of it. Even these consequences are of interest, as we shall explain—one of them is a non-trivial decomposition theorem of the first kind discussed above—but the method comes into its own when we introduce one or two further ideas in order to obtain conclusions that can be applied much more widely.

One of these ideas is the relatively standard one of *polynomial approximations*. Often we start with a function  $f$  that takes values in an interval  $[a, b]$ , and we want its structured part  $f_1$  to take values in  $[a, b]$  as well. If  $f_1$  is bounded, and if the class of structured functions is closed under composition with polynomials, then we can sometimes achieve this by choosing a polynomial  $P$  such that  $P(x)$  approximates  $a$  when  $x < a$ ,  $x$  when  $a \leq x \leq b$ , and  $b$  when  $x > b$ . Then the function  $Pf_1$  takes values in  $[a, b]$  (approximately), and under appropriate circumstances it is possible to argue that it approximates  $f_1$ . In section 4, we shall illustrate this technique by proving two results. The first is a fairly simple transference principle that we shall need later, and the second is a slightly more complicated version of it that is needed for proving the Green-Tao theorem. The latter is essentially the same as the “abstract structure theorem” of Tao and Ziegler [TZ], so called because it is an abstraction of arguments from the paper of Green and Tao. It is this second result that can be regarded as a major step in the proof of the Green-Tao theorem, and which is used to prove their transference principle. We shall end Section 4 with a brief description of the rest of the proof of Green and Tao.

In section 5, we shall prove the structure theorem of Tao mentioned earlier, and show how it leads to a strengthened decomposition theorem. We end the section, and the paper, with an indication of how to use the structure theorem.

## 2. SOME BASIC CONCEPTS OF ADDITIVE COMBINATORICS.

### 2.1. Preliminaries: Fourier transforms and $L_p$ -norms.

Let  $G$  be a finite Abelian group. A *character* on  $G$  is a non-zero function  $\psi : G \rightarrow \mathbb{C}$  with the property that  $\psi(xy) = \psi(x)\psi(y)$  for every  $x$  and  $y$ . It is easy to show that  $\psi$  must take values in the unit circle. It is also easy to show that two distinct characters are orthogonal. To see this, note first that if  $\psi_1$  and  $\psi_2$  are distinct, then  $\psi_1(\psi_2)^{-1}$  is a non-trivial character (that is, a character that is not identically 1). Next, note that if  $\psi$  is a non-trivial character and  $\psi(y) \neq 1$ , then  $\mathbb{E}_x \psi(x) = \mathbb{E}_x \psi(xy) = \psi(y) \mathbb{E}_x \psi(x)$ , so  $\mathbb{E}_x \psi(x) = 0$ .

(The notation “ $\mathbb{E}_x$ ” is shorthand for “ $|G|^{-1} \sum_{x \in G}$ ”.) This implies the orthogonality. Less obvious, but a straightforward consequence of the classification of finite Abelian groups, is the fact that the characters span all functions from  $G$  to  $\mathbb{C}$ : that is, they form an orthonormal basis of  $L_2(G)$ . (We shall discuss this space more in a moment.)

If  $f : G \rightarrow \mathbb{C}$ , then the *Fourier transform*  $\hat{f}$  of  $f$  tells us how to expand  $f$  in terms of the basis of characters. More precisely, one first defines the *dual group*  $\hat{G}$  to be the group of all characters on  $G$  under pointwise multiplication. Then  $\hat{f}$  is a function from  $\hat{G}$  to  $\mathbb{C}$ , defined by the formula

$$\hat{f}(\psi) = \mathbb{E}_x f(x) \overline{\psi(x)} = \mathbb{E}_x f(x) \psi(-x).$$

The *Fourier inversion formula* (which it is an easy exercise to verify) then tells us that

$$f(x) = \sum_{\psi} \hat{f}(\psi) \psi(x),$$

which gives the expansion of  $f$  as a linear combination of characters.

There are two natural measures that one can put on  $G$ : the uniform probability measure, and the counting measure (which assigns measure 1 to each singleton). Both of these are useful. The former is useful when one is looking at functions that are “flat”: an example would be the characteristic function of a dense subset  $A \subset G$ . If we write  $A(x)$  for  $\chi_A(x)$ , then  $\mathbb{E}_x A(x) = |G|^{-1} \sum_x A(x) = |A|/|G|$  is the density of  $A$ . The counting measure is more useful for functions  $F$  that are of “essentially bounded support”, in the sense that there is a set  $K$  of bounded size such that  $F$  is approximately equal (in some appropriate sense) to its restriction to  $K$ .

If  $f$  is a flat function, then there is a useful sense in which its Fourier transform is of essentially bounded support in the dual group. Therefore, if we are interested in flat functions defined on  $G$ , then we look at the uniform probability measure on  $G$  and the counting measure on the dual group  $\hat{G}$ . We then define inner products,  $L_p$ -norms, and  $\ell_p$ -norms as follows.

The inner product of two functions  $f$  and  $g$  from  $G$  to  $\mathbb{C}$  is the quantity  $\langle f, g \rangle = \mathbb{E}_x f(x) \overline{g(x)}$ . The resulting Euclidean norm is  $\|f\|_2 = \left( \mathbb{E}_x |f(x)|^2 \right)^{1/2}$ , and the Euclidean space is  $L_2$ . More generally,  $L_p$  is the space of all functions from  $G$  to  $\mathbb{C}$ , with the norm  $\|f\|_p = \left( \mathbb{E}_x |f(x)|^p \right)^{1/p}$ , where this is interpreted as  $\max |f(x)|$  when  $p = \infty$ .

On the dual group  $\hat{G}$  we have the same definitions, but with expectations replaced by sums. Thus,  $\langle F_1, F_2 \rangle = \sum_x F_1(x) \overline{F_2(x)}$  and  $\|F\|_p = \left( \sum_x |F(x)|^p \right)^{1/p}$ . The resulting space is denoted  $\ell_p$ . Once again,  $\|F\|_\infty$  is  $\max |F(x)|$ , so  $L_\infty$  and  $\ell_\infty$  are in fact the same space.

Two fundamental identities that are used repeatedly in additive combinatorics are the *convolution identity* and *Parseval's identity*. The *convolution*  $f * g$  of two functions  $f, g : G \rightarrow \mathbb{C}$  is defined by the formula

$$f * g(x) = \mathbb{E}_{y+z=x} f(y)g(z),$$

and the convolution identity states that  $(f * g)^\wedge(\psi) = \hat{f}(\psi)\hat{g}(\psi)$  for every  $\psi \in \hat{G}$ . That is, the Fourier transform “converts convolution into pointwise multiplication”. It also does the reverse: the Fourier transform of the pointwise product  $fg$  is the convolution  $\hat{f} * \hat{g}$ , where the latter is defined by the formula

$$\hat{f} * \hat{g}(\psi) = \sum_{\rho\sigma=\psi} \hat{f}(\rho)\hat{g}(\sigma).$$

Parseval's identity is the simple statement that  $\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle$ . It is important to keep in mind that the two inner products are defined differently, one with expectations and the other with sums, just as the convolutions were defined differently in  $G$  and  $\hat{G}$ . Setting  $f = g$  in Parseval's identity, we deduce that  $\|f\|_2 = \|\hat{f}\|_2$ .

The group that will interest us most is the cyclic group  $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$ . If we set  $\omega = \exp(2\pi i/N)$ , then any function of the form  $x \mapsto \omega^{rx}$  is a character, and the functions  $\omega^{rx}$  and  $\omega^{sx}$  are distinct if and only if  $r$  and  $s$  are not congruent mod  $N$ . Therefore, one can identify  $\mathbb{Z}_N$  with its dual, writing

$$\hat{f}(r) = \mathbb{E}_x f(x) \omega^{-rx}$$

whenever  $r$  is an element of  $\mathbb{Z}_N$ . However, the measure we use on  $\mathbb{Z}_N$  is different when we are thinking of it as a dual group.

The reason that Fourier transforms are important in additive combinatorics is that many quantities that arise naturally can be expressed in terms of convolutions, which can then be simplified by the Fourier transform. For instance, as we shall see in the next subsection, the quantity  $\mathbb{E}_{x,d} f(x)f(x+d)f(x+2d)$  arises naturally when one looks at arithmetic progressions of length 3. This can be rewritten as  $\mathbb{E}_{x,z} f(x)f(z)f((x+z)/2) = \mathbb{E}_{x,z} f(x)f(z)\overline{g(x+z)}$ , where  $g(u) = \overline{f(u/2)}$ . (We need  $N$  to be odd for this to make sense.) This is the inner product of  $f * f$  with  $g$ , so it is equal to  $\langle \hat{f}^2, \hat{g} \rangle$ .

## 2.2. What is additive combinatorics about?

The central objects of study in additive combinatorics are finite subsets of Abelian groups. For example, one of the main results in the area, Szemerédi's theorem, can be formulated as follows.

**Theorem 2.1.** *For every  $\delta > 0$  and every positive integer  $k$  there exists  $N$  such that every subset  $A \subset \mathbb{Z}_N$  of cardinality at least  $\delta N$  contains an arithmetic progression of length  $k$ .*

Here,  $\mathbb{Z}_N$  stands for the cyclic group  $\mathbb{Z}/N\mathbb{Z}$  of integers mod  $N$ , and an arithmetic progression of length  $k$  means a set of the form  $\{x, x + d, \dots, x + (k - 1)d\}$  with  $d \neq 0$ .

What are the *maps* of interest between finite subsets of Abelian groups? An initial guess might be that they were restrictions of group homomorphisms, but that turns out to be far too narrow a definition. Instead, they are functions called Freiman homomorphisms. A *Freiman homomorphism of order  $k$*  between sets  $A$  and  $B$  is a function  $\phi : A \rightarrow B$  such that

$$\phi(a_1) + \phi(a_2) + \dots + \phi(a_k) = \phi(a_{k+1}) + \phi(a_{k+2}) + \dots + \phi(a_{2k})$$

whenever

$$a_1 + a_2 + \dots + a_k = a_{k+1} + a_{k+2} + \dots + a_{2k}.$$

In particular, a Freiman homomorphism of order 2, often just known as a Freiman homomorphism, is a function such that  $\phi(a_1) + \phi(a_2) = \phi(a_3) + \phi(a_4)$  whenever  $a_1 + a_2 = a_3 + a_4$ . This is equivalent to the same definition with minus instead of plus, which is often more convenient.

A *Freiman isomorphism of order  $k$*  is a Freiman homomorphism of order  $k$  with an inverse that is also a Freiman homomorphism of order  $k$ . The rough idea is that a Freiman homomorphism of order  $k$  preserves all the linear structure of a set  $A$  that can be detected by integer combinations with coefficients adding up to 0 and with absolute values adding up to at most  $2k$ . For example, it is an easy exercise to show that if  $A$  is an arithmetic progression and  $B$  is Freiman-isomorphic to  $A$ , then  $B$  is also an arithmetic progression. This is because a sequence  $(x_1, x_2, \dots, x_m)$  is an arithmetic progression, written out in a sensible order, if and only if  $x_{i+2} - x_{i+1} = x_{i+1} - x_i$  for every  $i$ . It is also easy to show that if  $A$  and  $B$  are isomorphic, then their sumsets  $A + A$  and  $B + B$  have the same size.

Thus, a more precise description of the main objects studied by additive combinatorics would be that they are finite subsets of Abelian groups, up to Freiman isomorphisms of various orders.

An important aspect of results such as Szemerédi's theorem is that they have a certain "robustness". For instance, combining Szemerédi's theorem with a simple averaging argument, one can deduce the following corollary (which was first noted by Varnavides [V]).

**Corollary 2.2.** *For every  $\delta > 0$  and every positive integer  $k$  there exists  $\epsilon > 0$  such that, for every sufficiently large positive integer  $N$ , every subset  $A \subset \mathbb{Z}_N$  of cardinality at least  $\delta N$  contains at least  $\epsilon N^2$  arithmetic progressions of length  $k$ .*

A second important aspect is that they have "functional versions". One can regard a subset of  $\mathbb{Z}_N$  as a function that takes values 0 and 1. It turns out that many of the arguments used to prove Szemerédi's theorem apply to a much wider class of functions. In particular, they apply to functions that take values in the interval  $[0, 1]$ . The following generalization of Szemerédi's theorem is easily seen to follow from Corollary 2.2.

**Corollary 2.3.** *For every  $\delta > 0$  and every positive integer  $k$  there exists  $\epsilon > 0$  such that, for every positive integer  $N$  and every function  $f : \mathbb{Z}_N \rightarrow [0, 1]$  for which  $\mathbb{E}_x f(x) \geq \delta$ , we have the inequality*

$$\mathbb{E}_{x,d} f(x)f(x+d)\dots f(x+(k-1)d) \geq \epsilon.$$

Here,  $\mathbb{E}_{x,d}$  denotes the expectation over all pairs  $(x, d) \in \mathbb{Z}_N^2$ . It can be regarded as shorthand for  $N^{-2} \sum_{x,d}$ , but it is better to think in probabilistic terms: the left-hand side of the above inequality is then an expectation over all arithmetic progressions of length  $k$  (including degenerate ones with  $d = 0$ , but for large  $N$  these make a tiny contribution to the total).

A third important aspect is a deeper form of robustness. It turns out that quantities such as  $\mathbb{E}_{x,d} f(x)f(x+d)\dots f(x+(k-1)d)$  are left almost unchanged if you perturb  $f$  by adding a function  $g$  that is small in an appropriate norm. Furthermore, it is possible for  $g$  to be small in this norm even when the average size  $\mathbb{E}_x |g(x)|$  of  $g(x)$  is large: a typical example of such a function is one that takes the values  $\pm 1$  independently at random. The changes to the values of  $f$  are then quite large, but the randomness of  $g$  forces their contribution to expressions such as  $\mathbb{E}_{x,d} f(x)f(x+d)\dots f(x+(k-1)d)$  to cancel out almost completely. This cancellation, rather than smallness of a more obvious kind, is what justifies our thinking of  $f + g$  as a "perturbation" of  $f$ .

Thus, it is tempting to revise further our rough definition of additive combinatorics and say that the central objects of study are subsets of Abelian groups, up to Freiman isomorphism and "quasirandom perturbation". However, it takes some effort to make this idea precise, since the notion of a Freiman homomorphism does not apply as well to

functions as it does to sets (because it is insufficiently robust). Also, not every quantity of importance in the area is approximately invariant up to quasirandom perturbations: an example of one that isn't is the size of the sumset  $A + A$  of a set  $A$  of size  $n$ . So we shall content ourselves with the observation that all the results of this paper *are* approximately invariant.

So that we can say what this means, let us give some examples of norms that measure quasirandomness.

### 2.3. Uniformity norms for subsets of finite Abelian groups.

Let  $G$  be a finite Abelian group, and let  $g : G \rightarrow \mathbb{C}$ . The  $U^2$ -norm of  $g$  is defined by the formula

$$\|g\|_{U^2}^4 = \mathbb{E}_{x,a,b} g(x) \overline{g(x+a)g(x+b)} g(x+a+b).$$

We shall not give here the verification that this is a norm (though it will follow from a remark we make in 2.5), since our main concern is the sense in which it measures quasirandomness. It can be shown that if  $f : G \rightarrow \mathbb{C}$  is a function with  $\|f\|_\infty \leq 1$  and  $g$  is another such function with the additional property that  $\|g\|_{U^2}$  is small, then

$$\mathbb{E}_{x,d} f(x) f(x+d) f(x+2d) \approx \mathbb{E}_{x,d} (f+g)(x) (f+g)(x+d) (f+g)(x+2d).$$

A case of particular interest is when  $f$  is the characteristic function of a subset  $A \subset G$  of density  $\delta$ , which again we shall denote by  $A$ , and  $g(x) = A(x) - \delta$  for every  $x$ . If  $\|g\|_{U^2}$  is small, then we can think of  $f$  as a quasirandom perturbation of the constant function  $\delta$ . Then  $\mathbb{E}_{x,d} A(x) A(x+d) A(x+2d)$  will be around  $\delta^3$ , the approximate value it would take (with high probability) if the elements of  $A$  were chosen independently at random with probability  $\delta$ . When  $\|g\|_{U^2}$  is small, we say that  $A$  is a *quasirandom* subset of  $G$ . (This definition is essentially due to Chung and Graham [CG].)

In many respects, a quasirandom set behaves as one would expect a random set to behave, but in by no means all. For example, even if  $A$  is as quasirandom as it is possible for a set to be, it does not follow that

$$\mathbb{E}_{x,d} A(x) A(x+d) A(x+2d) A(x+3d) \approx \delta^4.$$

An example that shows this is the subset  $A \subset \mathbb{Z}_N$  that consists of all  $x$  such that  $x^2 \in [-\delta N/2, \delta N/2]$ . It can be shown that the density of  $A$  is very close to  $\delta$  when  $N$  is large,

and that  $\|g\|_{U^2} = \|A - \delta\|_{U^2}$  is extremely small. However, for this set  $A$ ,

$$\mathbb{E}_{x,d} A(x)A(x+d)A(x+2d)A(x+3d)$$

turns out to be at least  $c\delta^3$  for some absolute constant  $c > 0$ . We will not prove this here (a proof can be found in [G3]), but we give the example in order to draw attention to its quadratic nature. It turns out that this feature of the example is necessary, though quite what that means is not obvious, and the proof is even less so. See subsection 2.6 for further discussion of this.

This example shows that the smallness of the  $U^2$ -norm is not sufficient to explain all the typical behaviour of a random function. For this one needs to introduce “higher” uniformity norms, of which the next one is (unsurprisingly) the  $U^3$ -norm. If  $g$  is a function, then  $\|g\|_{U^3}^8$  is given by the expression

$$\mathbb{E}_{x,a,b,c} g(x)\overline{g(x+a)g(x+b)g(x+c)}g(x+a+b)g(x+a+c)g(x+b+c)\overline{g(x+a+b+c)}.$$

From this it is easy to guess the definition of the  $U^k$  norm, but for completeness here is a formula for it:

$$\|g\|_{U^k}^{2^k} = \mathbb{E}_{x,a_1,\dots,a_k} \prod_{\epsilon \in \{0,1\}^k} C^{|\epsilon|} g\left(x + \sum \epsilon_i a_i\right),$$

where  $C$  denotes the operation of complex conjugation and  $|\epsilon|$  denotes the number of non-zero coordinates of  $\epsilon$ .

These norms were introduced in [G1], where it was shown, as part of a proof of Szemerédi’s theorem, that if  $A$  is a subset of  $\mathbb{Z}_N$  of density  $\delta$ ,  $g(x) = A(x) - \delta$  for every  $x$ , and  $\|g\|_{U^k}$  is sufficiently small (meaning smaller than a positive constant that depends on  $\delta$  but not on  $N$ ), then

$$\mathbb{E}_{x,d} A(x)A(x+d)\dots A(x+kd) \approx \delta^{k+1}.$$

Let us call a set *uniform of degree  $k-1$*  if its  $U^k$ -norm is small. Then the above assertion is that a set of density  $\delta$  that is sufficiently uniform of degree  $k-1$  contains roughly as many arithmetic progressions  $(\bmod N)$  of length  $k+1$  as a random set of density  $\delta$  will (with high probability) contain. In particular, if  $A$  is *quadratically uniform* (meaning that the  $U^3$ -norm of  $A - \delta$  is sufficiently small), then

$$\mathbb{E}_{x,d} A(x)A(x+d)A(x+2d)A(x+3d) \approx \delta^4.$$

The arithmetic progression  $\{x, x+d, \dots, x+(k-1)d\}$  can be thought of as a collection of  $k$  linear forms in  $x$  and  $d$ . It can be shown that for any collection of linear forms in any

number of variables, there exists a  $k$  such that every set  $A$  that is sufficiently uniform of degree  $k$  contains about as many of the corresponding linear configurations as a random set of the same density. This was shown by Green and Tao [GT3], who generalized the argument in [G1]. The question of precisely which  $U^k$  norm is needed is a surprisingly subtle one. It is conjectured in [GW1] that the answer is the smallest  $k$  for which the  $k$ th powers of the linear forms in question are linearly independent. For instance, the configuration  $\{x, x+d, x+2d, x+3d\}$  needs the  $U^3$ -norm because  $x - 2(x+d) + (x+2d) = x^2 - 3(x+d)^2 + 3(x+2d)^2 - (x+3d)^2 = 0$ , but the cubes are linearly independent. A special case of this result is proved in [GW1] using “quadratic Fourier analysis”, which we will discuss in 2.7: to prove the full conjecture would require a theory of higher-degree Fourier analysis that will probably exist in due course but which has not yet been sufficiently developed.

#### 2.4. Uniformity norms for graphs and hypergraphs.

There are very close and important parallels between uniformity of subsets of finite Abelian groups, and quasirandomness of graphs and hypergraphs. For this reason, even though the relevant parts of graph and hypergraph theory belong to extremal combinatorics, they have become part of additive combinatorics as well: one could call them additive combinatorics without the addition.

Since that may seem a peculiar thing to say, let us briefly see what these parallels are. Let  $G$  be a graph on  $n$  vertices. One can think of  $G$  as a two-variable function  $G(x, y)$ , where  $x$  and  $y$  are vertices and  $G(x, y) = 1$  if  $xy$  is an edge and 0 otherwise. Just as we may regard a subset  $A$  of a finite Abelian group as quasirandom if a certain norm of  $A - \delta$  is small, we can regard a graph as quasirandom if a certain norm of the function  $G - \delta$  (where now  $\delta$  is the density  $\mathbb{E}_{x,y} G(x, y)$  of the graph  $G$ ) is small. This norm is given by the formula

$$\|g\|_{GU^2}^4 = \mathbb{E}_{x,x',y,y'} g(x, y) \overline{g(x, y')} g(x', y) \overline{g(x', y')},$$

which makes sense, and is useful, whenever  $X$  and  $Y$  are finite sets and  $g : X \times Y \rightarrow \mathbb{C}$ . The theory of quasirandom graphs was initiated by Thomason [Th] and more fully developed by Chung, Graham and Wilson [CGW]. The definition we have just given is equivalent to the definition in the latter paper.

To see how this relates to the  $U^2$  norm, let  $X$  and  $Y$  equal a finite Abelian group  $\Gamma$ , let  $f : \Gamma \rightarrow \mathbb{C}$  and let  $g(x, y) = f(x + y)$ . Then

$$\|g\|_{GU^2}^4 = \mathbb{E}_{x, x', y, y'} f(x + y) \overline{f(x + y')} \overline{f(x' + y)} f(x' + y').$$

The quadruples  $(x + y, x + y', x' + y, x' + y')$  are uniformly distributed over all quadruples  $(a, b, c, d)$  such that  $a + d = b + c$ . Since the same is true of all quadruples of the form  $(x, x + a, x + b, x + a + b)$ , we see that the right-hand side of the above formula is nothing other than  $\|f\|_{U^2}$ .

A similar argument can be used to relate the higher-degree uniformity norms to notions of quasirandomness for  $k$ -uniform *hypergraphs*, which are like graphs except that instead of having edges, which are pairs of vertices, one has hyperedges, which are  $k$ -tuples of vertices. The following formula defines a norm on  $k$ -variable functions:

$$\|g\|_{HU^k}^{2^k} = \mathbb{E}_{x_1^0, x_1^1} \dots \mathbb{E}_{x_k^0, x_k^1} \prod_{\epsilon \in \{0,1\}^k} C^{|\epsilon|} f(x_1^{\epsilon_1}, \dots, x_k^{\epsilon_k}).$$

If  $f(x_1, \dots, x_k)$  has the form  $g(x_1 + \dots + x_k)$ , then  $\|f\|_{HU^k} = \|g\|_{U^k}$ .

A hypergraph  $H$  of density  $\delta$  behaves in many respects like a random hypergraph of density  $\delta$  if  $\|H - \delta\|_{HU^k}$  is small enough. For instance, if  $k = 3$ , then the *simplex density*, which is given by the expression

$$\mathbb{E}_{x, y, z, w} H(x, y, z) H(x, y, w) H(x, z, w) H(y, z, w)$$

is roughly  $\delta^4$ , or what it would be in the random case. More generally, if  $\|H_i - \delta\|_{HU^k}$  is small for  $i = 1, 2, 3, 4$ , then

$$\mathbb{E}_{x, y, z, w} H_1(x, y, z) H_2(x, y, w) H_3(x, z, w) H_4(y, z, w)$$

is again roughly  $\delta^4$ . This assertion, which is proved by repeated use of the Cauchy-Schwarz inequality (see [G2] for a more general result), implies that

$$\mathbb{E}_{x, y, z, w} A(-3x - 2y - z) A(-2x - y + w) A(-x + z + 2w) A(y + 2z + 3w) \approx \delta^4$$

whenever  $A$  is a subset of  $\mathbb{Z}_N$  such that  $\|A - \delta\|_{U^3}$  is small. But the four linear forms above form an arithmetic progression of length 4 and common difference  $x + y + z + w$ : this is a sketch of what turns out to be the most natural proof that the  $U^3$ -norm controls arithmetic progressions of length 4.

These ideas can be developed to give a complete proof of Szemerédi's theorem: see [NRS], [RS], [G2], [T2].

## 2.5. Easy structure theorems for the $U^2$ -norm.

A great deal of information about the  $U^2$ -norm comes from the following simple observation.

**Lemma 2.4.** *Let  $G$  be a finite Abelian group and let  $f : G \rightarrow \mathbb{C}$ . Then  $\|f\|_{U^2} = \|\hat{f}\|_4$ .*

*Proof.* By the convolution identity and Parseval's identity,

$$\|f\|_{U^2}^4 = \mathbb{E}_{x+y=z+w} f(x)f(y)\overline{f(z)f(w)} = \langle f * f, f * f \rangle = \langle \hat{f}^2, \hat{f}^2 \rangle = \sum_{\psi} |\hat{f}(\psi)|^4.$$

The result follows on taking fourth roots.  $\square$

Now let us suppose that  $\|f\|_2 \leq 1$ , and let us fix some small constant  $\eta > 0$ . Then the number of characters  $\psi$  such that  $|\hat{f}(\psi)| \geq \eta$  is at most  $\eta^{-2}$ , since  $\sum_{\psi} |\hat{f}(\psi)|^2 = \|\hat{f}\|_2^2 = \|f\|_2^2 = 1$ . Using the inversion formula and this fact, we can split  $f$  into two parts,  $\sum_{\psi \in K} \hat{f}(\psi)\psi$  and  $\sum_{\psi \notin K} \hat{f}(\psi)\psi$ , where  $K$  is the set of all  $\psi$  such that  $|\hat{f}(\psi)| \geq \eta$ . Let us call these two parts  $g$  and  $h$ , respectively. The function  $g$  involves a bounded number of characters, and characters are functions that we can describe completely explicitly. Therefore, it can be thought of as the “structured” part of  $f$ . As for  $h$ , it is quasirandom, since

$$\|h\|_{U^2}^4 = \|\hat{h}\|_4^4 \leq \|\hat{h}\|_2^2 \|\hat{h}\|_{\infty}^2 \leq \eta^2 \|\hat{f}\|_2^2 \leq \eta^2.$$

Unfortunately, this simple decomposition turns out not to be very useful, for reasons that we shall explain later. We mention it in order to put some of our later results in perspective. The same applies to the next result, which shows that one can obtain a much stronger relationship between  $\|h\|_{U^2}$  and the upper bound on the size of  $K$  if one is prepared to tolerate a small  $L_2$ -error as well. This result and its proof are part of the standard folklore of additive combinatorics.

**Proposition 2.5.** *Let  $f$  be a function from a finite Abelian group  $G$  to  $\mathbb{C}$  and suppose that  $\|f\|_2 \leq 1$ . Let  $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a positive decreasing function that tends to 0 and let  $\epsilon > 0$ . Then there is a positive integer  $m$  such that  $f$  can be written as  $f_1 + f_2 + f_3$ , where  $f_1$  is a linear combination of at most  $m$  characters,  $\|f_2\|_{U^2} \leq \eta(m)$ , and  $\|f_3\|_2 \leq \epsilon$ .*

*Proof.* Let  $N = |G|$  and let us enumerate the dual group  $\hat{G}$  as  $\psi_1, \dots, \psi_N$  in such a way that the absolute values of the Fourier coefficients  $\hat{f}(\psi_i)$  are in non-increasing order. Choose an increasing sequence of positive integers  $m_1, m_2, \dots$  in such a way that  $m_{r+1} \geq \eta(m_r)^{-4}$  for every  $r$ .

Now let us choose  $i$  and attempt to prove the result using the decomposition  $f_1 = \sum_{i \leq m_r} \hat{f}(\psi_i)\psi_i$ ,  $f_2 = \sum_{i > m_{r+1}} \hat{f}(\psi_i)\psi_i$ , and  $f_3 = \sum_{m_r < i \leq m_{r+1}} \hat{f}(\psi_i)\psi_i$ . Then  $f_1$  is a linear

combination of at most  $m_r$  characters. Since there can be at most  $m_{r+1}$  characters  $\psi$  with  $|\hat{f}(\psi)| \geq m_{r+1}^{-1/2}$ , we find that

$$\|f_2\|_{U^2}^4 = \|\hat{f}_2\|_4^4 \leq m_{r+1}^{-1} \|\hat{f}_2\|_2^2 \leq \eta(m_r)^4 \|\hat{f}\|_2^2 \leq \eta(m_r)^4.$$

Therefore, we are done if  $\|f_3\|_2 \leq \epsilon$ . But the possible functions  $f_3$  (as  $r$  varies) are disjoint parts of the Fourier expansion of  $f$ , so at most  $\epsilon^{-2}$  of them can have norm greater than  $\epsilon$ . Therefore, we can find  $r \leq \epsilon^{-2}$  such that the proposed decomposition works.  $\square$

There is nothing to stop us taking  $m_1 = 1$ . The proof then gives us the desired decomposition for some  $m$  that is bounded above by a number that results from starting with 1 and applying the function  $t \mapsto \eta(t)^{-4}$  at most  $\epsilon^{-2}$  times. Although Proposition 2.5 is still not all that useful, it resembles other results that are, as we shall see in due course. In those results, it is common to require  $\eta(m)$  to be exponentially small: the resulting bound is then of tower type.

## 2.6. Inverse theorems.

A *direct theorem* in additive number theory is one that starts with a description of a set and uses that description to prove that the set has certain additive properties. For instance, the statement that every positive integer is the sum of four squares starts with the explicitly presented set  $S$  of all perfect squares, and proves that the four-fold sumset  $S + S + S + S$  is the whole of  $\mathbb{N} \cup \{0\}$ . An *inverse theorem* is a result that goes in the other direction: one starts with a set  $A$  that is assumed to have certain properties, and attempts to find some kind of description of  $A$  that explains those properties. Ideally, this description should be so precise that it actually characterizes the properties in question: a set  $A$  has the properties if and only if it satisfies the description.

A remarkable inverse theorem, which lies at the heart of many recent results in additive combinatorics, is a theorem of Freiman [F] (later given a considerably more transparent proof by Ruzsa [Ru]) that characterizes sets that have small sumsets. If  $A$  is a set of  $n$  integers, then it is easy to show that the sumset  $A + A$  has size at least  $2n - 1$  and at most  $n(n + 1)/2$ . What can be said about  $A$  if the size of the sumset is close to its minimum, in the sense that  $|A + A| \leq C|A|$  for some fixed constant  $C$ ? A simple example of such a set is an arithmetic progression. A slightly less simple example is a set  $A$  that is contained in an arithmetic progression of length at most  $Cn/2$ . A less simple example altogether is a “two-dimensional arithmetic progression”: that is, a set of the form  $\{x_0 + rd_1 + sd_2 : 0 \leq r < t_1, 0 \leq s < t_2\}$ . If  $A$  is such a set, then  $|A + A| \leq 4|A|$ , and

more generally if  $A$  is a  $k$ -dimensional arithmetic progression (the definition of which is easy to guess), then  $|A + A| \leq 2^k |A|$ . As in the one-dimensional case, one can pass to large subsets and obtain more examples. Freiman's theorem states that one has then exhausted all examples.

**Theorem 2.6.** *For every  $C$  there exist  $k$  and  $K$  such that every set  $A$  of  $n$  integers such that the sumset  $A + A$  has size at most  $Cn$  is contained in an arithmetic progression of dimension at most  $k$  and cardinality at most  $Kn$ .*

Freiman's theorem has been extremely influential, in large part because of Ruzsa's proof, which was extremely elegant and conceptual, and gave much better bounds than Freiman's argument. These bounds have subsequently been improved by Chang [C], who added further interesting ingredients to Ruzsa's argument. A generalization of Freiman's theorem to subsets of an arbitrary Abelian group was proved by Green and Ruzsa [GR].

The notion of an inverse theorem makes sense also for functions defined on Abelian groups. For instance, here is a simple inverse theorem about functions with large  $U^2$ -norm.

**Proposition 2.7.** *Let  $c > 0$ , let  $G$  be a finite Abelian group, let  $f : G \rightarrow \mathbb{C}$  be a function such that  $\|f\|_2 \leq 1$  and suppose that  $\|f\|_{U^2} \geq c$ . Then there exists a character  $\psi$  such that  $|\langle f, \psi \rangle| \geq c^2$ .*

*Proof.* By Lemma 2.4 and our assumptions about  $f$ ,

$$c^2 \leq \|\hat{f}\|_4^2 \leq \|\hat{f}\|_\infty \|\hat{f}\|_2 \leq \|\hat{f}\|_\infty,$$

which is what is claimed.  $\square$

Conversely, and without any assumption about  $\|f\|_2$ , if there exists a character  $\psi$  such that  $\langle f, \psi \rangle \geq c$ , then  $\|f\|_{U^2} = \|f\|_4 \geq c^{1/4}$ . Therefore, correlation with a character “explains” the largeness of the  $U^2$ -norm.

What about the  $U^3$ -norm? This turns out to be a much deeper question. As our remarks earlier have suggested, quadratic functions come into play when one starts to think about it. For example, if  $f : \mathbb{Z}_N \rightarrow \mathbb{C}$  is the function  $x \mapsto \omega^{2rx^2}$  for some  $r$  (where  $\omega$  is once again equal to  $\exp(2\pi i/N)$ ), then the identity

$$x^2 - (x+a)^2 - (x+b)^2 - (x+c)^2 + (x+a+b)^2 + (x+a+c)^2 + (x+b+c)^2 - (x+a+b+c)^2 = 0$$

implies easily that  $\|f\|_{U^3} = 1$ . However, it is also easy to show that  $f$  does not correlate significantly with any character. Therefore, we are forced to consider quadratic functions. If  $q$  is a quadratic function, then let us call the function  $\omega^q$  a *quadratic phase function*.

It is tempting to conjecture that a bounded function  $f$  with  $U^3$ -norm at least  $c$  must correlate with a quadratic phase function, meaning that  $\mathbb{E}_x f(x) \omega^{q(x)} \geq c'$  for some quadratic function  $q$  and some constant  $c'$  that depends on  $c$  only. However, although such a correlation is a sufficient condition for the  $U^3$ -norm of  $f$  to be large, it is not necessary, because there are “multidimensional” examples. For instance, if  $P$  is the two-dimensional arithmetic progression  $\{x_0 + rd_1 + sd_2 : 0 \leq r < t_1, 0 \leq s < t_2\}$ , then we can define something like a quadratic form  $q$  on  $P$  by the formula  $q(x_0 + rd_1 + sd_2) = ar^2 + brs + cs^2$ . We can then define a function  $f$  to be  $\omega^{q(x)}$  when  $x \in P$  and 0 otherwise. Let us call such a function a *generalized quadratic phase function*. It is not hard to prove that such functions have large  $U^3$  norms, and that they do not have to correlate with ordinary quadratic phase functions.

In [G1], the following “weak inverse theorem” was proved for all  $U^k$  norms.

**Theorem 2.8.** *Let  $c > 0$  be a constant and let  $f : \mathbb{Z}_N \rightarrow \mathbb{C}$  be a function such that  $\|f\|_\infty \leq 1$  and  $\|f\|_{U^k} \geq c$ . Then there is a partition of  $\mathbb{Z}_N$  into arithmetic progressions  $P_i$  of length at least  $N^{\alpha(c,k)}$ , and for each  $P_i$  there is a polynomial  $r_i$  of degree at most  $k$  such that, writing  $\pi_i$  for the density  $|P_i|/N$  of  $P_i$ , we have  $\sum_i \pi_i |\mathbb{E}_{x \in P_i} f(x) \omega^{r_i(x)}| \geq c/2$ .*

This result was the main step in the proof of Szemerédi’s theorem given in [G1]. The reason that this is a “weak inverse theorem” is that the converse is far from true. The result shows that  $f$  correlates with a function that is made out of many fragments of polynomial phase functions, but it does not provide what one might hope for: correlation with a single generalized polynomial phase function. However, the proof strongly suggested that such a result should be true, and Green and Tao, by adding some important further ingredients, have established a strong inverse theorem in the quadratic case [GT2]. Let us state their result a little imprecisely.

**Theorem 2.9.** *Let  $c > 0$  be a constant and let  $f : \mathbb{Z}_N \rightarrow \mathbb{C}$  be a function such that  $\|f\|_\infty \leq 1$  and  $\|f\|_{U^k} \geq c$ . Then there exists a constant  $c'$  that depends on  $c$  only, and a generalized quadratic phase function  $g$ , such that  $|\langle f, g \rangle| \geq c'$ .*

## 2.7. Higher Fourier analysis.

As we have seen, the  $U^2$ -norm of a function  $f$  is equal to the  $\ell_4$ -norm of its Fourier transform, and this observation leads quickly to a decomposition of functions into a structured part and a quasirandom part. Is there a comparable result for the  $U^3$ -norm? The inverse theorem of Green and Tao suggests that we should try to decompose  $f$  into generalized

quadratic phase functions. However, there are far more than  $N$  of these, so they do not form an orthonormal basis, or indeed a basis of any kind. One might nevertheless hope for some canonical way of decomposing a function, but it is far from clear that there is one—certainly, nobody has come close to finding one.

However, one can still hope for a structure theorem that resembles Proposition 2.5. We would expect it to say that a function  $f$  can be decomposed into a linear combination of a small number of generalized quadratic phase functions, plus a function with very small  $U^3$ -norm, plus a function that is small in  $L_2$ . Green and Tao deduced such a result from their inverse theorem, and thereby initiated a form of *quadratic Fourier analysis*. In [GW1], a different method was given for deducing somewhat different decomposition theorems from inverse theorems. The main ingredient of this method was the Hahn-Banach theorem: the proof will be sketched in the next section. This gave an alternative form of quadratic Fourier analysis, which provided much better bounds for the results of that paper (the ones that concerned controlling systems of linear forms with  $U^k$ -norms).

We shall have more to say about higher Fourier analysis later in the paper.

## 2.8. Easy structure theorems for graphs.

We have already seen that the  $U^2$ -norm of the one-variable function  $g$ , defined on a finite Abelian group  $G$ , can be regarded as the  $GU^2$ -norm of the two-variable function  $f(x, y) = g(x + y)$ . The relationship does not stop here, however. If  $\psi$  is a character, then, for any  $x$ ,

$$\mathbb{E}_y g(x + y)\psi(-y) = \psi(x)\mathbb{E}_y g(x + y)\psi(-x - y) = \hat{g}(\psi)\psi(x)$$

This shows that characters are similar to eigenvectors of the symmetric matrix  $f(x, y)$ , except that they are mapped to multiples of their complex conjugates. However, it is notable that the corresponding “eigenvalues” are the Fourier coefficients of the function  $g$ . This observation suggests, correctly as it turns out, that eigenvalues play a similar role for real symmetric matrices to the role played by Fourier coefficients for functions defined on finite Abelian groups.

We briefly illustrate this by proving a result that is analogous to Proposition 2.5. First, we prove a well-known lemma relating the  $GU^2$ -norm to eigenvalues. It will tie in better with our previous notation (and with applications of matrices to graph theory) if we use a slightly unconventional association between matrices and linear maps, as we did above. Given a matrix  $f(x, y)$  and a function  $u(y)$  we shall think of  $fu(x)$  as the quantity  $\mathbb{E}_y f(x, y)u(y)$  rather than the same thing with a sum.

The finite-dimensional spectral theorem tells us that a real symmetric matrix  $f(x, y)$  has an orthonormal basis of eigenvectors. If these are  $u_1, \dots, u_n$  and the corresponding eigenvalues are  $\lambda_1, \dots, \lambda_n$ , then we can express this by saying that

$$f(x, y) = \sum_i \lambda_i u_i \otimes u_i,$$

where  $u \otimes v$  denotes the function  $u(x)v(y)$ . To see why these are the same, consider the effect of each side in turn on a basis vector  $u_j$ . On the one hand, we have  $\mathbb{E}_y f(x, y)u_j(y) = \lambda_j u_j(x)$  (by our unconventional definition of matrix multiplication) while on the other we have

$$\mathbb{E}_y \sum_i \lambda_i u_i \otimes u_i(x, y)u_j(y) = \sum_i \lambda_i u_i(x) \mathbb{E}_y u_i(y)u_j(y) = \sum_i \lambda_i u_i(x) \delta_{ij} = \lambda_j u_j(x)$$

by the orthonormality (with respect to the  $L_2$ -norm) of the eigenvectors.

**Lemma 2.10.** *Let  $X$  be a finite set and let  $f$  be a symmetric real-valued function defined on  $X^2$ . Let the eigenvalues of  $f$  be  $\lambda_1, \dots, \lambda_n$ . Then  $\|f\|_{GU^2}^4 = \sum_r \lambda_r^4$ .*

*Proof.* All results of this kind are proved by expanding the expression for  $\|f\|_{GU^2}^4$  in terms of the spectral decomposition  $\sum_r \lambda_r u_r \otimes u_r$  of  $f$ .

$$\begin{aligned} \|f\|_{GU^2}^4 &= \mathbb{E}_{x,x'} \mathbb{E}_{y,y'} f(x, y) f(x, y') f(x', y) f(x', y') \\ &= \mathbb{E}_{x,x'} \mathbb{E}_{y,y'} \sum_{p,q,r,s} \lambda_p \lambda_q \lambda_r \lambda_s u_p(x) u_p(y) u_q(x) u_q(y') u_r(x') u_r(y) u_s(x') u_s(y') \\ &= \sum_{p,q,r,s} \lambda_p \lambda_q \lambda_r \lambda_s \delta_{pq} \delta_{pr} \delta_{rs} \delta_{qs} \\ &= \sum_r \lambda_r^4 \end{aligned}$$

as claimed.  $\square$

A similar but easier proof establishes that  $\|f\|_2^2 = \sum_r \lambda_r^2$ .

The next result is a direct analogue for symmetric two-variable real functions (and therefore in particular for graphs) of Proposition 2.5

**Proposition 2.11.** *Let  $X$  be a finite set and let  $f$  be a symmetric real-valued function on  $X^2$  such that  $\|f\|_2 \leq 1$ . Let  $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a positive decreasing function that tends to 0 and let  $\epsilon > 0$ . Then there is a positive integer  $m$  such that  $f$  can be written as  $f_1 + f_2 + f_3$ , where  $f_1$  is a linear combination of at most  $m$  orthonormal functions of the form  $u \otimes u$ ,  $\|f_2\|_{GU^2} \leq \eta(m)$ , and  $\|f_3\|_2 \leq \epsilon$ .*

*Proof.* Let  $N = |X|$  and let us enumerate an orthonormal basis  $(u_i)_1^N$  of eigenvectors of  $f$  in such a way that the absolute values of the eigenvalues  $\lambda_i$  are in non-increasing order. Choose an increasing sequence of positive integers  $m_1, m_2, \dots$  in such a way that  $m_{r+1} \geq \eta(m_r)^{-4}$  for every  $r$ .

Now let us choose  $i$  and attempt to prove the result using the decomposition  $f_1 = \sum_{i \leq m_r} \lambda_i u_i \otimes u_i$ ,  $f_2 = \sum_{i > m_{r+1}} \lambda_i u_i \otimes u_i$ , and  $f_3 = \sum_{m_r < i \leq m_{r+1}} \lambda_i u_i \otimes u_i$ . Then  $f_1$  is a linear combination of at most  $m_r$  eigenvectors, which are orthonormal to each other by the spectral theorem. By the remark above about the sum of the squares of the eigenvalues, there can be at most  $m_{r+1}$  eigenvectors  $u_i$  with  $|\lambda_i| \geq m_{r+1}^{-1/2}$ . Therefore,

$$\|f_2\|_{U^2}^4 = \sum_{i > m_{r+1}} \lambda_i^4 \leq m_{r+1}^{-1} \sum_i \lambda_i^2 = m_{r+1}^{-1} \|f\|_2^2 \leq \eta(m_r)^4.$$

Therefore, we are done if  $\|f_3\|_2 \leq \epsilon$ . But the possible functions  $f_3$  (as  $r$  varies) are disjoint parts of the spectral expansion of  $f$ , so at most  $\epsilon^{-2}$  of them can have norm greater than  $\epsilon$ . Therefore, we can find  $r \leq \epsilon^{-2}$  such that the proposed decomposition works.  $\square$

The above result is closely related to a “weak regularity lemma” due to Frieze and Kannan [FK].

### 3. THE HAHN-BANACH THEOREM AND SIMPLE APPLICATIONS.

Let us begin by stating the version of the Hahn-Banach theorem that we shall need.

**Theorem 3.1.** *Let  $K$  be a convex body in  $\mathbb{R}^n$  and let  $f$  be an element of  $\mathbb{R}^n$  that is not contained in  $K$ . Then there is a constant  $\beta$  and a non-zero linear functional  $\phi$  such that  $\langle f, \phi \rangle \geq \beta$  and  $\langle g, \phi \rangle \leq \beta$  for every  $g \in K$ .*

Now let us prove two corollaries, both of which are useful for proving decomposition theorems.

**Corollary 3.2.** *Let  $K_1, \dots, K_r$  be closed convex subsets of  $\mathbb{R}^n$ , each containing 0, let  $c_1, \dots, c_r$  be positive real numbers and suppose that  $f$  is an element of  $\mathbb{R}^n$  that cannot be written as a sum  $f_1 + \dots + f_r$  with  $f_i \in c_i K_i$ . Then there is a linear functional  $\phi$  such that  $\langle f, \phi \rangle > 1$  and  $\langle g, \phi \rangle \leq c_i^{-1}$  for every  $i \leq r$  and every  $g \in K_i$ .*

*Proof.* Let  $K$  be the convex body  $\sum_i c_i K_i$ . Our hypothesis is that  $f \notin K$ . Since  $K$  is closed, it follows that there exists  $\epsilon > 0$  such that  $(1 + \epsilon)^{-1} f \notin K$ . Therefore, by Theorem 3.1, there is a constant  $\beta$  and a linear functional  $\phi$  such that  $(1 + \epsilon)^{-1} \langle f, \phi \rangle \geq \beta$  and  $\langle g, \phi \rangle \leq \beta$  for every  $g \in K$ . Again using the fact that  $K$  is closed, we can add a small

Euclidean ball  $B$  to  $K$  in such a way that  $(1 + \epsilon)^{-1}f \notin B + K$ . Since  $0 \in K$ , it follows that  $\beta > 0$ . Therefore, we can divide  $\phi$  by  $\beta$  and get  $\beta$  to be 1, with the result that  $\langle f, \phi \rangle \geq (1 + \epsilon)\beta$ . Since 0 belongs to each  $K_i$ , we can also conclude that  $\langle g, \phi \rangle \leq 1$  for every  $g \in c_i K_i$ , which completes the proof.  $\square$

**Corollary 3.3.** *Let  $K_1, \dots, K_r$  be closed convex subsets of  $\mathbb{R}^n$ , each containing 0 and suppose that  $f$  is an element of  $\mathbb{R}^n$  that cannot be written as a convex combination  $c_1 f_1 + \dots + c_r f_r$  with  $f_i \in K_i$ . Then there is a linear functional  $\phi$  such that  $\langle f, \phi \rangle > 1$  and  $\langle g, \phi \rangle \leq 1$  for every  $i \leq r$  and every  $g \in K_i$ .*

*Proof.* Let  $K$  be the set of all convex combinations  $c_1 f_1 + \dots + c_r f_r$  with  $f_i \in K_i$ . Then  $K$  is a closed convex set and  $f$  is not contained in  $K$ . Therefore, there exists  $\epsilon > 0$  such that  $(1 + \epsilon)^{-1}f \notin K$ . By Theorem 3.1 there is a functional  $\phi$  and a constant  $\beta$  such that  $(1 + \epsilon)^{-1}\langle f, \phi \rangle \geq \beta$  and  $\langle g, \phi \rangle \leq \beta$  whenever  $g$  belongs to  $K$ . In particular,  $\langle g, \phi \rangle \leq \beta$  whenever  $g$  belongs to one of the sets  $K_i$ . As in the proof of the previous corollary,  $\beta$  must be positive and can therefore be assumed to be 1. The result follows.  $\square$

Recall that if  $\|\cdot\|$  is a norm on  $\mathbb{R}^n$ , then the dual norm  $\|\cdot\|^*$  is defined by the formula  $\|\phi\|^* = \max\{\langle f, \phi \rangle : \|f\| \leq 1\}$ . If  $f \in \mathbb{R}^n$  then Theorem 3.1 implies that there exists a functional  $\phi$  such that  $\|\phi\|^* \leq 1$  and  $\langle f, \phi \rangle = \|f\|$ . Such a functional is called a *support functional* for  $f$ . In this paper it will be convenient to call  $\phi$  a support functional if  $\phi \neq 0$  and  $\langle f, \phi \rangle = \|f\|\|\phi\|^*$ , so that a positive scalar multiple of a support functional is also a support functional.

The following lemma is useful in proofs that involve the Hahn-Banach theorem, as we shall see in section 3.2. It tells us that the dual of an  $\ell_1$ -like combination of norms is an  $\ell_\infty$ -like combination of their duals. We shall adopt the convention that if  $\|\cdot\|$  is a norm defined on a subspace  $V$  of  $\mathbb{R}^n$  then its dual  $\|\cdot\|^*$  is the seminorm defined by the formula  $\|f\|^* = \max\{\langle f, g \rangle : g \in V, \|g\| \leq 1\}$ .

**Lemma 3.4.** *Let  $\Sigma$  be a set and for each  $\sigma \in \Sigma$  let  $\|\cdot\|_\sigma$  be a norm defined on a subspace  $V_\sigma$  of  $\mathbb{R}^n$ . Suppose that  $\sum_{\sigma \in \Sigma} V_\sigma = \mathbb{R}^n$ , and define a norm  $\|\cdot\|$  on  $\mathbb{R}^n$  by the formula*

$$\|x\| = \inf\{\|x_1\|_{\sigma_1} + \dots + \|x_k\|_{\sigma_k} : x_1 + \dots + x_k = x, \sigma_1, \dots, \sigma_k \in \Sigma\}$$

*Then this formula does indeed define a norm, and its dual norm  $\|\cdot\|^*$  is given by the formula*

$$\|z\|^* = \max\{\|z\|_\sigma^* : \sigma \in \Sigma\}$$

*Proof.* It is a simple exercise to check that the expression does indeed define a norm.

Let us begin by supposing that  $\|z\|_\sigma^* \geq 1$  for some  $\sigma \in \Sigma$ . Then there exists  $x \in V_\sigma$  such that  $\|x\|_\sigma \leq 1$  and  $|\langle x, z \rangle| \geq 1$ . But then  $\|x\| \leq 1$  as well, from which it follows that  $\|z\|^* \geq 1$ . Therefore,  $\|z\|^*$  is at least the maximum of the  $\|z\|_\sigma^*$ .

Now let us suppose that  $\|z\|^* > 1$ . This means that there exists  $x$  such that  $\|x\| \leq 1$  and  $|\langle x, z \rangle| \geq 1 + \epsilon$  for some  $\epsilon > 0$ . Let us choose  $x_1, \dots, x_k$  such that  $x_i \in V_{\sigma_i}$  for each  $i$ ,  $x_1 + \dots + x_k = x$ , and  $\|x_1\|_{\sigma_1} + \dots + \|x_k\|_{\sigma_k} < 1 + \epsilon$ . Then

$$\sum_i |\langle x_i, z \rangle| > \|x_1\|_{\sigma_1} + \dots + \|x_k\|_{\sigma_k}$$

so there must exist  $i$  such that  $|\langle x_i, z \rangle| > \|x_i\|_{\sigma_i}$ , from which it follows that  $\|z\|_i^* > 1$ . This proves that  $\|z\|^*$  is at most the maximum of the  $\|z\|_i^*$ .  $\square$

A particular case that will interest us is when  $\Sigma$  is a subset of  $\mathbb{R}^n$ , for each  $\sigma \in \Sigma$  the subspace  $V_\sigma$  is just the subspace generated by  $\sigma$ , and the norm on  $V_\sigma$  is  $\|\lambda\sigma\|_\sigma = |\lambda|$ . The dual seminorm is then  $\|f\|_\sigma^* = |\langle f, \sigma \rangle|$ . Thus, if we specialize Lemma 3.4 to this case then we obtain the following corollary.

**Corollary 3.5.** *Let  $\Sigma \subset \mathbb{R}^n$  be a set that spans  $\mathbb{R}^n$  and define a norm  $\|\cdot\|$  on  $\mathbb{R}^n$  by the formula*

$$\|f\| = \inf \left\{ \sum_{i=1}^k |\lambda_i| : f = \sum_{i=1}^k \lambda_i \sigma_i, \sigma_1, \dots, \sigma_k \in \Sigma \right\}.$$

*Then this formula does indeed define a norm, and its dual norm  $\|\cdot\|^*$  is defined by the formula  $\|f\|^* = \max\{|\langle f, \sigma \rangle| : \sigma \in \Sigma\}$ .*

### 3.1. A simple structure theorem.

We now prove a very simple (and known) decomposition result that illustrates our basic method.

**Proposition 3.6.** *Let  $\|\cdot\|$  be any norm on  $\mathbb{R}^n$  and let  $f$  be any function in  $\mathbb{R}^n$ . Then  $f$  can be written as  $g + h$  in such a way that  $\|g\| + \|h\|^* \leq \|f\|_2$ .*

*Proof.* Suppose that the result is false. We shall apply Corollary 3.3 to the function  $f/\|f\|_2$ , with  $K_1$  and  $K_2$  taken to be the unit balls of  $\|\cdot\|$  and  $\|\cdot\|^*$ . Our hypothesis is equivalent to the assertion that  $f/\|f\|_2$  is not a convex combination  $c_1 g_1 + c_2 g_2$  with  $g_i \in K_i$  for  $i = 1, 2$ . Therefore, we obtain a functional  $\phi$  such that  $\langle f, \phi \rangle > \|f\|_2$  and  $\|\phi\|^*$  and  $\|\phi\|$  are both at most 1. But the first property implies, by the Cauchy-Schwarz inequality, that  $\|\phi\|_2 > 1$ , while the second implies that  $\|\phi\|_2^2 = \langle \phi, \phi \rangle \leq \|\phi\| \|\phi\|^* \leq 1$ . This is a contradiction.  $\square$

A simple modification of Proposition 3.6 makes it a little more flexible. Suppose that we wish to write  $f$  as  $g + h$  with  $\|g\|$  small and  $\|h\|^*$  not too large. If we define a new norm  $|\cdot|$  to be  $\epsilon^{-1}\|\cdot\|$ , then  $|\cdot|^* = \epsilon\|\cdot\|^*$ . Applying Proposition 3.6 to these rescaled norms, we find that we can write  $f$  as  $g + h$  in such a way that  $\epsilon^{-1}\|g\| + \epsilon\|h\|^* \leq \|f\|_2$ . In particular, if  $\|f\|_2 = 1$ , then  $\|g\| \leq \epsilon$  and  $\|h\|^* \leq \epsilon^{-1}$ .

The reason such a result might be expected to be useful in additive combinatorics is that, as demonstrated in the previous section, we have a good supply of norms  $\|\cdot\|$  that measure quasirandomness. Moreover, their duals, as we shall see later, can be thought of as a sort of measure of structure. Perhaps the simplest example that illustrates this is if we look at functions  $f$  defined on finite Abelian groups, and take  $\|f\|$  to be  $\|\hat{f}\|_\infty$ . If  $\|f\|_2 \leq 1$ , then

$$\|f\|_{U^2}^2 = \|\hat{f}\|_4^2 \leq \|\hat{f}\|_2 \|\hat{f}\|_\infty \leq \|\hat{f}\|_\infty,$$

a calculation we have already done. This shows that for functions with bounded  $L_2$ -norm there is a rough equivalence between  $\|f\|_{U^2}$  and  $\|\hat{f}\|_\infty$ , in the sense that if one is small then so is the other.

Thus, if  $\|f\|_2 \leq 1$  then  $\|f\| = \|\hat{f}\|_\infty$  being small tells us that  $f$  is quasirandom. The dual norm,  $\|f\|^* = \|\hat{f}\|_1$ , is a sort of measure of structure, since if  $\|\hat{f}\|_1$  is at most  $C$ , then  $f$  is a small multiple of a convex combination of trigonometric functions, which can be approximated in  $L_2$  by a linear combination of a bounded number of such functions. Thus, we recover a result that resembles Proposition 2.5. It is weaker, however, because we have not yet related the quasirandomness constant to the structure constant by means of an arbitrary function. However, this is easily done, again with the help of an  $L_2$  error term, as the next result shows.

**Proposition 3.7.** *Let  $f$  be a function in  $\mathbb{R}^n$  with  $\|f\|_2 \leq 1$  and let  $\|\cdot\|$  be any norm on  $\mathbb{R}^n$ . Let  $\epsilon > 0$  and let  $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be any decreasing positive function. Let  $r = \lceil 2\epsilon^{-1} \rceil$  and define a sequence  $C_1, \dots, C_r$  by setting  $C_1 = 1$  and  $C_i = 2\eta(C_{i-1})^{-1}$  when  $i > 1$ . Then there exists  $i \leq r$  such that  $f$  can be decomposed as  $f_1 + f_2 + f_3$  with*

$$C_i^{-1}\|f_1\|^* + \eta(C_i)^{-1}\|f_2\| + \epsilon^{-1}\|f_3\|_2 \leq 1.$$

In particular,  $\|f_1\|^* \leq C_i$ ,  $\|f_2\| \leq \eta(C_i)$  and  $\|f_3\|_2 \leq \epsilon$ .

*Proof.* If there is no such decomposition for  $i$ , then by Corollary 3.3 there is a functional  $\phi_i$  such that  $\|\phi_i\| \leq C_i^{-1}$ ,  $\|\phi_i\|^* \leq \eta(C_i)^{-1}$ ,  $\|\phi_i\|_2 \leq \epsilon^{-1}$ , and  $\langle \phi_i, f \rangle > 1$ . If this is true for every  $i \leq r$  then

$$\|\phi_1 + \dots + \phi_r\|_2 \geq \langle \phi_1 + \dots + \phi_r, f \rangle \geq r,$$

where the first inequality follows from Cauchy-Schwarz and the assumption that  $\|f\|_2 \leq 1$ .

On the other hand, if  $i < j$  then

$$\langle \phi_i, \phi_j \rangle \leq \|\phi_i\| \|\phi_j\|^* \leq \eta(C_i)^{-1} C_j^{-1} \leq 1/2,$$

the last inequality following from the way we constructed the sequence  $C_1, \dots, C_r$ . Therefore,

$$\|\phi_1 + \dots + \phi_r\|_2^2 \leq \epsilon^{-1} r + r(r-1)/2.$$

This contradicts the previous estimate, since  $r \geq 2\epsilon^{-1}$ .  $\square$

It is easy to deduce Proposition 2.5 from this result. Of course, this fact on its own is not a very convincing demonstration of the utility of the Hahn-Banach theorem, since for the norm  $\|f\| = \|\hat{f}\|_\infty$  it is easy to write down an explicit decomposition of  $f$ , as we saw in section 2.5. But there are other important norms where this is certainly not the case. For example, if we take  $\|f\|$  to be  $\|f\|_{U^k}$  for a larger  $k$ , then there is no obvious decomposition of  $f$  from which we can read off three functions  $f_1, f_2$  and  $f_3$  with the required properties.

### 3.2. Deducing decomposition theorems from inverse theorems.

The main result of [GW1] shows that certain linear configurations occur with the “expected” frequency in any set  $A$  for which the balanced function  $A - \delta$  (where  $\delta$  is the density of  $A$ ) has sufficiently small  $U^2$ -norm. The interest in the result is that the  $U^2$ -norm suffices for the configurations in question, whereas the natural arguments that generalize the proof that the  $U^k$  norm controls progressions of length  $k - 2$  would suggest that the  $U^3$ -norm was needed. In order to prove the result, a form of quadratic Fourier analysis was needed, as we have already mentioned. The approach in [GW1] was to apply directly a result of Green and Tao, which obtains a decomposition of a bounded function  $f$  by constructing an averaging projection  $P$  with the property that  $f - Pf$  has small  $U^3$  norm. However, there was a technicality involved that forced us to use an iterated version of their result that gives rise to very weak bounds. In order to obtain reasonable bounds for the problem, it turned out to be convenient—indeed, as far as we could tell, necessary—to prove a decomposition theorem that could be regarded as a quadratic analogue of Proposition 2.5, with the important difference that the strong dependence of  $\eta(m)$  on  $m$  was not needed. (This was why it was possible to obtain good bounds.)

The argument appears in [GW2], and it can be regarded as a special case of a general principle that can be informally summarized as follows: *to each inverse theorem there is*

a corresponding decomposition theorem. It is possible to give a formal statement, as will be clear from our discussion, but in practice it is much easier to describe a *method* for deducing decompositions from inverse theorems than it is to state an artificial lemma that declares that the method works. The main reason for this is that when one applies the method, one typically starts with the decomposition one wants to prove and the inverse theorem one *can* prove, and adjusts the former until it follows from the latter. We shall reflect this in our discussion below: more precisely, we shall assume that a decomposition of a certain general kind does *not* exist, draw an easy consequence from this, and see when this consequence contradicts any given inverse theorem.

Suppose, then, that we have a subset  $\Sigma \subset \mathbb{R}^n$  of functions that we regard as “structured”, and suppose that the functions in  $\Sigma$  span  $\mathbb{R}^n$ . Suppose also that we have another function  $f$  that we would ideally like to decompose as a linear combination  $\sum_{i=1}^k \lambda_i \sigma_i$  of functions  $\sigma_i \in \Sigma$  with  $\sum_{i=1}^k |\lambda_i|$  not too large, together with some error terms. That is, we look for a result of the following kind.

**Hoped-for decomposition.** *The function  $f$  can be written in the form*

$$f = \sum_{i=1}^k \lambda_i \sigma_i + g_1 + \cdots + g_r,$$

where  $\sum_{i=1}^k |\lambda_i| \leq M$ , each  $\sigma_i$  belongs to  $\Sigma$ , and for each  $j \leq r$  we have an inequality of the form  $\|g_j\|_{(j)} \leq \eta_j$ .

Typically,  $r$  will be a very small integer such as 2.

Lemma 3.4 says that the formula

$$\begin{aligned} \|g\| &= \inf \left\{ \sum_{i=1}^k |\lambda_i| : g = \sum_{i=1}^k \lambda_i \sigma_i, \sigma_i \in \Sigma \right\} \\ &= \inf \left\{ \sum_{i=1}^k \|g\|_{\sigma_i} : g = g_1 + \cdots + g_k, \sigma_1, \dots, \sigma_k \in \Sigma, g_i \in V_{\sigma_i} \right\} \end{aligned}$$

defines a norm, and that the dual of this norm is the norm

$$\|\phi\|^* = \max_{\sigma \in \Sigma} |\langle \sigma, \phi \rangle|.$$

Now let us suppose that no decomposition of the kind we are looking for exists. This is equivalent to the assumption that  $f$  has no decomposition of the form  $g_0 + g_1 + \cdots + g_k$  with  $\|g_0\| \leq M$  and  $\|g_i\|_{(i)} \leq \eta_i$  for every  $i$ . If this is the case, then by Corollary 3.2 there

must be a linear functional  $\phi$  such that  $\langle f, \phi \rangle > 1$ ,  $\|\phi\|^* \leq M^{-1}$ , and  $\|\phi\|_{(i)}^* \leq \eta_i^{-1}$  for  $i = 1, 2, \dots, r$ .

The statement that  $\|\phi\|^* \leq M^{-1}$  tells us that  $|\langle \sigma, \phi \rangle| \leq M^{-1}$  for every  $\sigma \in \Sigma$ . Thus, what we would like is an inverse theorem that concludes the opposite: that there must be some  $\sigma \in \Sigma$  such that  $|\langle \sigma, \phi \rangle| > M^{-1}$ . Before we think about this, let us list the assumptions that we have at our disposal.

**Consequences of failure of decomposition.** *Suppose that there is no decomposition  $f = \sum_{i=1}^k \lambda_i \sigma_i + g_1 + \dots + g_r$  such that  $\sum_{i=1}^k |\lambda_i| \leq M$ , each  $\sigma_i$  belongs to  $\Sigma$ , and  $\|g_j\|_{(j)} \leq \eta_j$  for each  $j \leq r$ . Then there exists  $\phi$  such that*

- (i)  $\langle \sigma, \phi \rangle \leq M^{-1}$  for every  $\sigma \in \Sigma$ ;
- (ii)  $\langle f, \phi \rangle > 1$ ;
- (iii)  $\|\phi\|_{(j)}^* \leq \eta_j^{-1}$  for  $j = 1, 2, \dots, r$ .

The assumptions of an inverse theorem are typically that  $f$  is not too big in one norm, such as, for instance, the  $L_\infty$ -norm, but not too small in another, such as the  $U^3$ -norm. The only information we have that could possibly imply a lower bound on any norm of  $\phi$  is the inequality  $\langle f, \phi \rangle > 1$ , and even that does not help unless we have an *upper* bound on some norm of  $f$ . (Of course, it is hardly surprising that such a bound would be required for a theorem that allows us to decompose  $f$  into a bounded combination of bounded functions.)

So let us suppose that we have an inverse theorem of the following form. (We have introduced the constant  $K$  to allow us to multiply  $\phi$  by an arbitrary non-zero scalar.)

**Putative Inverse Theorem.** *Let  $\phi \in \mathbb{R}^n$  be a function such that  $\|\phi\| \leq K$  and  $\|\|\phi\|\| \geq \epsilon$ . Then there exists  $\sigma \in \Sigma$  such that  $|\langle \sigma, \phi \rangle| \geq Kc(\epsilon/K)$ .*

This will be contradicted under the following circumstances:

- (a) the upper bounds  $\|\phi\|_{(i)}^* \leq \eta_i^{-1}$  imply that  $\|\phi\| \leq K$ ;
- (b) the upper bounds on the  $\|\phi\|_{(i)}^*$ , an upper bound on some norm of  $f$ , and the lower bound  $\langle f, \phi \rangle > 1$ , together imply that  $\|\|\phi\|\| \geq \epsilon$ ;
- (c)  $M^{-1} < Kc(\epsilon/K)$ .

For example, suppose that  $M^{-1} < Kc(\epsilon\eta)$  and we would like a decomposition  $f = \sum_{i=1}^k \lambda_i \sigma_i + g + h$  with  $\sum_{i=1}^k |\lambda_i| \leq M$ ,  $\|g\| \leq \epsilon$  and  $\|h\|^* \leq \eta$ . If such a decomposition does not exist, then we obtain  $\phi$  such that  $\langle \sigma, \phi \rangle < \eta^{-1}c(\epsilon\eta)$  for every  $\sigma \in \Sigma$ ,  $\|\|\phi\|\|^* \leq \epsilon^{-1}$ ,  $\|\phi\| \leq \eta^{-1}$ , and  $\langle f, \phi \rangle > 1$ . If we also know that  $\|f\|_2 \leq 1$ , then it follows that  $\|\phi\|_2 \geq 1$ . But since  $\|\phi\|_2^2 \leq \|\|\phi\|\| \cdot \|\|\phi\|\|^*$ , it follows that  $\|\|\phi\|\| \geq \epsilon$ . This contradicts the inverse theorem (with  $K = \eta^{-1}$ ).

If we know a little bit more about  $f$ , then we can obtain a correspondingly stronger result. For instance, suppose that we know that  $\|f\|^* \leq \epsilon^{-1}$ . Then the bound  $\langle f, \phi \rangle > 1$  immediately implies that  $\|\phi\| > \epsilon$ , so we do not need the error term  $g$  in the decomposition.

Decomposition results obtained by the simple argument above—just assume that a decomposition doesn't exist, apply Hahn-Banach, and contradict an inverse theorem—can be very useful. However, in order to use them one has to do a little more work. For example, it is not usually trivial that a sum of the form  $\sum_{i=1}^k \lambda_i \sigma_i$  is “structured”, even if the sum  $\sum_{i=1}^k |\lambda_i|$  is smallish and all the individual functions  $\sigma_i$  are highly structured. The difficulty is that  $k$  may be very large, and in order to deal with it one tends to need a principle that says that functions  $\sigma_i$  are either “closely related” or “far apart”. A simple example is when  $\Sigma$  is the set of all characters, in which case any two elements of  $\Sigma$  are either identical or orthogonal. In [GW2] a lemma was proved to the effect that two generalized quadratic phases were either “linearly related” or “approximately orthogonal”. That made it possible to replace the linear combination by a much smaller linear combination of slightly more general functions.

A second point is that one sometimes wants more information about the “structured function”  $f_1 = \sum_{i=1}^k \lambda_i \sigma_i$ . For instance, if  $\|f\|_\infty \leq 1$  it can be extremely helpful to know that  $\|f_1\|_\infty \leq 1$  as well. This does not come directly out of the method above, but it does when we combine that method with methods that we shall discuss in the next section.

Just before we finish this section, we observe that inverse theorems can be used to prove strengthened decomposition theorems as well: that is, ones where some of the  $\eta_i$  can be made to depend on  $M$ . Suppose, for example, that our inverse theorem tells us that whenever  $\|\phi\|_\infty \leq 1$  and  $\|\phi\| \geq \epsilon$  there must exist  $\sigma \in \Sigma$  such that  $|\sigma, \phi| \geq c(\epsilon)$ . Suppose also that (as often happens)  $\|f\|^* \geq \|f\|_\infty$  for every  $f \in \mathbb{R}^n$ . Now let  $f$  be a function with  $\|f\|_2 \leq 1$  and use Proposition 2.5 to write  $f$  as  $f_1 + f_2 + f_3$  with  $\|f_1\|^* \leq C$ ,  $\|f_2\| \leq \eta(C)$  and  $\|f_3\|_2 \leq \theta$ . In our discussion just after the statement of the putative inverse theorem, we observed that knowing that  $\|f_1\|^* \leq C$  would yield a decomposition  $f_1 = \sum_{i=1}^k \lambda_i \sigma_i + h$ , where  $\sum_{i=1}^k |\lambda_i| \leq c(\theta C^{-2})$  (taking  $\epsilon = C^{-1}$ ,  $K = C$ , and replacing  $\eta$  by  $\theta$ ), and  $\|h\|_1 \leq \theta$ . Therefore, we can decompose  $f$  as  $\sum_{i=1}^k \lambda_i \sigma_i + f_2 + f_3 + h$ , with  $\sum_{i=1}^k |\lambda_i| \leq c(\theta C^{-2})$ ,  $\|f_2\| \leq \eta(C)$ , and  $\|f_3 + h\|_1 \leq 2\theta$ . Since  $\eta$  is an arbitrary function, we can make it depend in an arbitrary way on  $c(\theta C^{-2})$ . Thus, we have obtained the following result. (Note that the constants and functions are not the same as the constants and functions with the same names in the discussion that has just finished.)

**Theorem 3.8.** *Let  $\Sigma$  be a subset of  $\mathbb{R}^n$  that spans  $\mathbb{R}^n$ . Let  $\|\cdot\|$  be a norm such that  $\|f\|_\infty \leq \|f\|^*$  for every  $f \in \mathbb{R}^n$ . Suppose that for every function  $f$  with  $\|f\|_\infty \leq 1$  and  $\|f\| \geq \epsilon$  there exists  $\sigma \in \Sigma$  such that  $|\langle f, \sigma \rangle| \geq c(\epsilon)$ . Let  $\theta > 0$  and let  $\eta$  be a decreasing function from  $\mathbb{R}_+$  to  $\mathbb{R}_+$ . Then there exists a constant  $C_0$ , depending on  $\eta$  and  $\theta$  only, such that every function  $f \in \mathbb{R}^n$  with  $\|f\|_2 \leq 1$  has a decomposition*

$$f = \sum_{i=1}^k \lambda_i \sigma_i + f_2 + f_3$$

*with the following property: each  $\sigma_i$  belongs to  $\Sigma$  and there is a constant  $C \leq C_0$  such that  $\sum_{i=1}^k |\lambda_i| \leq C$ ,  $\|f_2\| \leq \eta(C)$ , and  $\|f_3\|_1 \leq \eta$ .*

#### 4. THE POSITIVITY AND BOUNDEDNESS PROBLEMS.

Although our results so far are sometimes useful, they have a serious limitation. Suppose, for example, that we wish to use Proposition 3.7. What we would like to do is use the structural properties of  $h$  to prove that certain quantities, such as  $\mathbb{E}_{x,d} h(x)h(x+d)h(x+2d)$ , are large, and then to show that  $f = g + h$  is a “random enough” perturbation of  $h$  for  $\mathbb{E}_{x,d} f(x)f(x+d)f(x+2d)$  to be large as well. But even if  $h$  is a small linear combination of just a few trigonometric functions, there is no particular reason for  $\mathbb{E}_{x,d} h(x)h(x+d)h(x+2d)$  to be large. If we want it to be large, then we need additional assumptions. The most useful one in practice is *positivity*.

Suppose that  $f \in \mathbb{R}^n$  is a function with  $\|f\|_2 \leq 1$  and that it takes non-negative values. With an appropriate choice of norm  $\|\cdot\|$ , Proposition 3.7 allows us to decompose  $f$  into a “structured part”, a “quasirandom part” and a small  $L_2$  error. One’s intuition suggests that the structured part of a non-negative function should not need to take negative values, and this turns out to be correct for the norms discussed in section 2.3.

In section 5 we shall prove a very general result of this kind. In this section, we shall prove some simpler results that illustrate the method of polynomial approximations; we shall use this method repeatedly later.

##### 4.1. Algebra norms, polynomial approximation and a first transference theorem.

To begin with, we need a definition that will pick out the class of norms for which we can prove results. Actually, for now we shall give a definition that is not always broad enough

to be useful. In the next section we shall define a broader class of norms to which the method still applies.

**Definition.** Let  $X$  be a finite set. An algebra norm on  $\mathbb{R}^X$  is a norm  $\|\cdot\|$  such that  $\|fg\| \leq \|f\|\|g\|$  for any two functions  $f$  and  $g$ , and  $\|\mathbf{1}\| = 1$ .

A good example of an algebra norm—indeed, the central example—is the  $\ell_1$ -norm of the Fourier transform of  $f$ , which has the submultiplicativity property because

$$\|\widehat{fg}\|_1 = \|\widehat{f} * \widehat{g}\|_1 \leq \|\widehat{f}\|_1 \|\widehat{g}\|_1.$$

The predual of this norm (it is of course the dual as well but we shall be thinking of it as the primary norm and the algebra norm as its dual) is the  $\ell_\infty$  norm of  $\widehat{f}$ , which, as we have already seen, is in a crude sense equivalent to the  $U^2$ -norm for many functions of interest.

We shall use the following simple lemma repeatedly.

**Lemma 4.1.** Let  $\|\cdot\|$  be a norm on  $\mathbb{R}^n$  such that the dual norm  $\|\cdot\|^*$  is an algebra norm. Then  $\|f\| \geq |\mathbb{E}_x f(x)|$  and  $\|f\|^* \geq \|f\|_\infty$  for every function  $f$ .

*Proof.* Since  $\|\cdot\|^*$  is an algebra norm,  $\|\mathbf{1}\|^* = 1$ , so  $\|f\| \geq |\langle f, \mathbf{1} \rangle| = |\mathbb{E}_x f(x)|$ .

For the second part, if  $\|f\|^* \leq 1$  then  $\|f^n\|^* \leq 1$  for every  $n$ . It follows that  $\|f\|_\infty \leq 1$ , since otherwise at least one coordinate of  $f_n$  would be unbounded. Therefore,  $\|f\|_\infty \leq \|f\|^*$  for every  $f$ .  $\square$

The Weierstrass approximation theorem tells us that every continuous function on a closed bounded interval can be uniformly approximated by polynomials. It will be helpful to define a function connected with this result. Given a real polynomial  $P$ , let  $R_P$  be the polynomial obtained from  $P$  by replacing all the coefficients of  $P$  by their absolute values. If  $J : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function,  $C$  is a positive real number and  $\delta > 0$ , let  $\rho(C, \delta, J)$  be twice the infimum of  $R_P(C)$  over all polynomials  $P$  such that  $|P(x) - J(x)| \leq \delta$  for every  $x \in [-C, C]$ . So that it will not be necessary to remember the definition of  $\rho(C, \delta, J)$  we now state and prove a simple but very useful lemma.

**Lemma 4.2.** Let  $\|\cdot\|^*$  be an algebra norm, let  $J : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function and let  $C$  and  $\delta$  be positive real numbers. Then there exists a polynomial  $P$  such that  $\|P\phi - J\phi\|_\infty \leq \delta$  and  $\|P\phi\|^* \leq \rho(C, \delta, J)$  for every  $\phi \in \mathbb{R}^n$  such that  $\|\phi\|^* \leq C$ .

*Proof.* It is immediate from the definition of  $\rho(C, \delta, J)$  that for every  $C$  and every  $\delta > 0$  there exists a polynomial  $P$  such that  $|P(x) - J(x)| \leq \delta$  for every  $x \in [-C, C]$ , and such that  $R_P(C) \leq \rho(C, \delta, J)$ .

Now let  $\phi \in \mathbb{R}^n$  be a function with  $\|\phi\|^* \leq C$ . Then  $\|\phi\|_\infty \leq C$  as well, since  $\|\cdot\|^*$  is an algebra norm. Since  $P$  and  $J$  agree to within  $\delta$  on  $[-C, C]$ , it follows that  $\|P\phi - J\phi\|_\infty \leq \delta$ .

Suppose that  $P$  is the polynomial  $P(x) = a_n x^n + \cdots + a_1 x + a_0$ . Then, by the triangle inequality and the algebra property of  $\|\cdot\|^*$ ,

$$\begin{aligned}\|P\phi\| &\leq |a_n| \|\phi^n\|^* + \cdots + |a_1| \|\phi\|^* + |a_0| \\ &\leq |a_n| (\|\phi\|^*)^n + \cdots + |a_1| \|\phi\|^* + |a_0| \\ &= R_P(\|\phi\|^*).\end{aligned}$$

Since the coefficients of  $R_P$  are all non-negative, this is at most  $R_P(C)$ , which is at most  $\rho(C, \delta, J)$ , by our choice of  $P$ .  $\square$

In more qualitative terms, the above lemma tells us that if  $\phi$  is bounded in an algebra norm and we compose it with an arbitrary continuous function  $J$ , then the resulting function  $J\phi$  can be uniformly approximated by functions that are still bounded in the algebra norm.

The next result is our first transference theorem of the paper. It tells us that if  $\mu$  and  $\nu$  are non-negative functions on a set  $X$  and they are sufficiently close in an appropriate norm, then any non-negative function that is dominated by  $\mu$  can be “transferred to”—that is, approximated by—a non-negative function that is dominated by  $\nu$ . We shall apply this principle in Section 5. As we shall see later in this section, it is also not hard to generalize the result to obtain a generalized version of the Green-Tao transference theorem.

**Theorem 4.3.** *Let  $\mu$  and  $\nu$  be non-negative functions on a set  $X$  and suppose that  $\|\mu\|_1$  and  $\|\nu\|_1$  are both at most 1. Let  $\eta, \delta > 0$ , let  $J : \mathbb{R} \rightarrow \mathbb{R}$  be the function given by  $J(x) = (x + |x|)/2$  and let  $\epsilon = \delta/2\rho(\eta^{-1}, \delta/4, J)$ . Let  $\|\cdot\|$  be a norm on  $\mathbb{R}^X$  such that the dual norm  $\|\cdot\|^*$  is an algebra norm and suppose that  $\|\mu - \nu\| \leq \epsilon$ . Then for every function  $f$  with  $0 \leq f \leq \mu$  there exists a function  $g$  such that  $0 \leq g \leq \nu(1 - \delta)^{-1}$  and  $\|f - g\| \leq \eta$ .*

*Proof.* An equivalent way of stating the conclusion is that  $f = g + h$  with  $0 \leq g \leq \nu(1 - \delta)^{-1}$  and  $\|h\| \leq \eta$ . Thus, if the result is false then we can find a functional  $\phi$  such that  $\langle f, \phi \rangle > 1$ , but  $\langle g, \phi \rangle \leq 1$  for every  $g$  such that  $0 \leq g \leq \nu(1 - \delta)^{-1}$ , and  $\|\phi\|^* \leq \eta^{-1}$ .

The first condition on  $\phi$  is equivalent to the statement that  $\langle \nu, \phi_+ \rangle \leq 1 - \delta$ . To see this, note that for any  $\phi$ , the  $g$  that maximizes  $\langle g, \phi \rangle$  takes the value 0 when  $\phi(x) < 0$  and  $\nu(x)(1 - \delta)^{-1}$  when  $\phi(x) \geq 0$ , in which case  $\langle g, \phi \rangle = (1 - \delta)^{-1} \langle \nu, \phi_+ \rangle$ .

Now  $\phi_+$  is equal to  $J\phi$ . Since  $\|\cdot\|^*$  is an algebra norm, we can apply Lemma 4.2 and obtain a polynomial  $P$  such that  $\|P\phi - \phi_+\|_\infty \leq \delta/4$  and  $\|P\phi\|^* \leq R_P(C) = \rho(\eta^{-1}, \delta/4, J)$ , which we shall abbreviate to  $\rho$ .

Since  $\langle \nu, \phi_+ \rangle \leq 1 - \delta$  and  $\|\nu\|_1 \leq 1$ , it follows that  $\langle \nu, P\phi \rangle \leq 1 - 3\delta/4$ . Since  $\|P\phi\|^* \leq \rho$  and  $\|\mu - \nu\| \leq \epsilon$ , it follows that  $\langle \mu, P\phi \rangle \leq 1 - 3\delta/4 + \epsilon\rho$ . Since  $\|\mu\|_1 \leq 1$ , it follows that  $\langle \mu, \phi_+ \rangle \leq 1 - \delta/2 + \epsilon\rho$ . Since  $f \leq \mu$  it follows that  $\langle f, \phi_+ \rangle \leq 1 - \delta/2 + \epsilon\rho$ , and since  $f \geq 0$  it follows that  $\langle f, \phi \rangle \leq 1 - \delta/2 + \epsilon\rho$ , which is a contradiction.  $\square$

#### 4.2. Approximate duality and algebra-like structures.

As the previous section shows, we can carry out polynomial-approximation arguments when we are looking at a norm  $\|\cdot\|$  for which the dual norm  $\|\cdot\|^*$  is an algebra norm. A key insight of Green and Tao (which has received less comment than other aspects of their proof) is that one can carry out polynomial-approximation arguments under hypotheses that are weaker in two respects: one can use pairs of norms that are not precisely dual to each other, and the norm that measures structure can have much weaker properties than those of an algebra norm. It is not hard to generalize the arguments in an appropriate way: the insight was to see that there were important situations in which one could obtain the weaker hypotheses even when the stronger ones were completely false.

To see why this might be, think once again about the one algebra norm we have so far considered, namely  $\|\hat{f}\|_\infty$ . For bounded functions  $f$ , this is closely related (by Proposition 2.7 and the remark after it) to  $\|\hat{f}\|_4$ , which equals the  $U^2$ -norm, so we can deduce facts related to the  $U^2$ -norm from the fact that  $\|\hat{f}\|_1$  is an algebra norm.

We can regard this argument as carrying out the following procedure. First, we establish an inverse theorem for the  $U^2$ -norm: this is what we did in Proposition 2.7. We then note that the functions that we obtain in the inverse theorem, namely the characters, are closed under pointwise multiplication. And then we make the following observation.

**Lemma 4.4.** *Let  $X$  be a set of functions in  $\mathbb{C}^n$  that spans all of  $\mathbb{C}^n$ , contains the constant function  $\mathbf{1}$ , and is closed under pointwise multiplication. Suppose also that  $\|\phi\|_\infty \leq 1$  for every function  $\phi \in X$ . Then the norm  $\|\cdot\|$  on  $\mathbb{R}^n$  defined by the formula*

$$\|f\| = \inf \left\{ \sum_{i=1}^k |\lambda_i| : f_1, \dots, f_k \in X, f = \sum_{i=1}^k \lambda_i f_i \right\}$$

*is an algebra norm.*

*Proof.* Suppose that  $f = \sum_{i=1}^k \lambda_i f_i$  and  $g = \sum_{j=1}^l \mu_j g_j$ , with all  $f_i$  and  $g_j$  in  $X$ . Then  $fg = \sum_{i=1}^k \sum_{j=1}^l \lambda_i \mu_j f_i g_j$ . Since  $X$  is closed under pointwise multiplication, each  $f_i g_j$  belongs to  $X$ . Moreover,  $\sum_{i=1}^k \sum_{j=1}^l |\lambda_i| |\mu_j| = \sum_{i=1}^k |\lambda_i| \sum_{j=1}^l |\mu_j|$ . From this the submultiplicativity follows easily. The fact that  $\|\mathbf{1}\| = 1$  follows from the assumption that  $\mathbf{1} \in X$  and that all functions in  $X$  have  $L_\infty$ -norm at most 1.  $\square$

In the case where  $X$  is the set of all characters on a finite Abelian group, the norm given by Lemma 4.4 is the  $\ell_1$ -norm of the Fourier transform.

Now suppose that we want to prove comparable facts about the  $U^3$ -norm. An obvious approach would be to use Theorem 2.9, the inverse theorem for the  $U^3$ -norm. However, the generalized quadratic phase functions that appear in the conclusion of that theorem are not quite closed under pointwise multiplication: associated with them are certain parameters that one wants to be small, which obey rules such as  $\gamma(fg) \leq \gamma(f) + \gamma(g)$ .

As we shall see, this is not a serious difficulty, because often one can restrict attention to products of a bounded number of functions that an inverse theorem provides. A more fundamental problem is that for the higher  $U^k$ -norms we do not (yet) have an inverse theorem. Or at least, we do not have an inverse theorem where the function that appears in the conclusion can be explicitly described. What Green and Tao did to get round this difficulty was to define a class of functions that they called basic anti-uniform functions, and to prove a “soft” inverse theorem concerning those functions.

**Definition.** For every function  $f \in \mathbb{R}^n$ , let  $\mathcal{D}f$  be the function defined by the formula

$$\mathcal{D}f(x) = \mathbb{E}_{a,b,c} f(x+a)f(x+b)f(x+c)\overline{f(x+a+b)f(x+a+c)f(x+b+c)}f(x+a+b+c).$$

Let  $X$  be a subset of  $\mathbb{R}^n$ . A basic anti-uniform function (with respect to  $X$ ) is a function of the form  $\mathcal{D}f$  with  $f \in X$ .

Needless to say, the above definition generalizes straightforwardly to a class of basic anti-uniform functions for the  $U^k$ -norm, for any  $k$ . The same applies to the next proposition.

**Proposition 4.5.** Let  $X$  be a subset of  $\mathbb{R}^n$  and let  $f \in X$  be a function such that  $\|f\|_{U^3} \geq \epsilon$ . Then there is a basic anti-uniform function  $g$ , with respect to  $X$ , such that  $\langle f, g \rangle \geq \epsilon^8$ .

*Proof.* The way we have stated the result is rather artificial, since the basic anti-uniform function in question is nothing other than  $\mathcal{D}f$ . Moreover, it is trivial that  $\langle f, \mathcal{D}f \rangle \geq \epsilon^8$ , since if we expand the left-hand side we obtain the formula for  $\|f\|_{U^3}^8$ .  $\square$

Of course, the price one pays for such a simple proof is that one has far less information about basic anti-uniform functions than one would have about something like a generalized polynomial phase function. In particular, it is not obvious what one can say about products of basic anti-uniform functions.

We remark here that an inequality proved in [G1] implies easily that  $\langle g, \mathcal{D}f \rangle \leq \|g\|_{U^3} \|f\|_{U^3}^7$  for every function  $g$ . Thus,  $\|\mathcal{D}f\|_{U^3}^* \leq \|f\|_{U^3}^7$ . Since  $\langle f, \mathcal{D}f \rangle = \|f\|_{U^3} \|f\|_{U^3}^7$ , we see that  $\mathcal{D}f$  is a support functional for  $f$ . It is not hard to show that it is unique (up to a scalar multiple). Since every function is a support functional for something, it may seem as though there is something odd about the definition of a basic anti-uniform function. However, it is less all-encompassing than it seems, because we are restricting attention to functions  $\mathcal{D}f$  for which  $f$  belongs to some specified class of functions  $X$ . (Nevertheless, we shall usually choose  $X$  in such a way that every function is a multiple of a basic anti-uniform function.)

A crucial fact that Green and Tao proved about basic anti-uniform functions is that, for suitable sets  $X$ , their products have  $(U^k)^*$ -norms that can be controlled. To be precise, they proved the following lemma. (It is not stated as a lemma, but rather as the beginning step in the proof of their Lemma 6.3.)

**Lemma 4.6.** *For every positive integer  $K$  there is a constant  $C_K$  such that if  $\mathcal{D}f_1, \dots, \mathcal{D}f_K$  are basic anti-uniform functions [with respect to a suitable set  $X$ ], then  $\|\mathcal{D}f_1 \dots \mathcal{D}f_K\|_{U^k}^* \leq C_K$ .*

### 4.3. A generalization of the Green-Tao-Ziegler transference theorem.

We shall be more concerned with the form of Lemma 4.6 than with the details of what  $X$  is, since our aim is to describe in an abstract way the important properties of the operator  $f \mapsto \mathcal{D}f$ . This we do in the next definition, which is meant to capture the idea that the dual of a certain norm somewhat resembles an algebra norm.

**Definition.** *Let  $\|\cdot\|$  be a norm on  $\mathbb{R}^n$  such that  $\|f\|_\infty \leq \|f\|^*$  for every  $f \in \mathbb{R}^n$ , and let  $X$  be a bounded subset of  $\mathbb{R}^n$ . Then  $\|\cdot\|$  is a quasi algebra predual norm, or QAP-norm, with respect to  $X$  if there is a (non-linear) operator  $\mathcal{D} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  a strictly decreasing function  $c : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , and an increasing function  $C : \mathbb{N} \rightarrow \mathbb{R}$  with the following properties:*

- (i)  $\langle f, \mathcal{D}f \rangle \leq 1$  for every  $f \in X$ ;
- (ii)  $\langle f, \mathcal{D}f \rangle \geq c(\epsilon)$  for every  $f \in X$  with  $\|f\| \geq \epsilon$ ;
- (iii)  $\|\mathcal{D}f_1 \dots \mathcal{D}f_K\|^* \leq C(K)$  for any functions  $f_1, \dots, f_K \in X$ .

It will help to explain the terminology if we introduce another norm, which we shall call  $\|\cdot\|_{BAC}$ . It is given by the formula  $\|f\|_{BAC} = \max\{|\langle f, \mathcal{D}g \rangle| : g \in X\}$ . The letters “BAC” stand for “basic anti-uniform correlation” here. We shall call the functions  $\mathcal{D}f$  with  $f \in X$  basic anti-uniform functions, and assume for convenience that they span  $\mathbb{R}^n$ , so that  $\|\cdot\|_{BAC}$  really is a norm. Of course, this norm depends on  $X$  and  $\mathcal{D}$ , but we are suppressing the dependence in the notation.

By Lemma 3.5 the dual of the norm  $\|\cdot\|_{BAC}$  is given by the formula

$$\|f\|_{BAC}^* = \inf\left\{\sum_{i=1}^k |\lambda_i| : f = \sum_{i=1}^k \lambda_i \mathcal{D}f_i, f_1, \dots, f_k \in X\right\}.$$

Thus, it measures the ease with which a function can be decomposed into a linear combination of basic anti-uniform functions. In terms of this norm, property (ii) above is telling us that if  $f \in X$  and  $\|f\| \geq \epsilon$  then  $\|f\|_{BAC} \geq c(\epsilon)$ . This expresses a rough equivalence between the two norms, of a similar kind to the rough equivalence between  $\|f\|_{U^2}$  and  $\|\hat{f}\|_\infty$  when  $\|f\|_\infty \leq 1$ . It can also be thought of as a soft inverse theorem; property (iii) then tells us that the functions that we obtain from this inverse theorem have products that are not too big.

Now let us briefly see why Theorem 4.3 generalizes easily from preduals of algebra norms to QAP-norms. The following result is not quite the result alluded to in the title of this subsection, but it *is* an abstract principle that can be used as part of the proof of the Green-Tao theorem. As we mentioned in the introduction, the proof given here is much shorter and simpler than the proof given by Green and Tao. (This is not quite trivial to verify as they do not explicitly state the result, but the proof here can be used to simplify Section 6 of their paper slightly, and to replace Sections 7 and 8 completely.)

As a first step, we shall generalize Lemma 4.2, the simple result about polynomial approximations. The generalization is equally straightforward: the main difference is merely that we need a modification of the definition of the polynomial  $R_P$ . Let us suppose that  $\|\cdot\|$  is a QAP-norm and let  $C : \mathbb{N} \rightarrow \mathbb{R}$  be the function given in property (iii) of that definition. If  $P$  is the polynomial  $p(x) = a_n x^n + \dots + a_1 x + a_0$ , then we define  $R'_P$  to be the polynomial  $C(n)|a_n|x^n + \dots + C(1)|a_1|x + |a_0|$ : that is, we replace the  $k$ th coefficient of  $P$  by its absolute value and multiply it by  $C(k)$ . If  $J : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function, and  $C_1, C_2$  and  $\delta$  are positive real numbers, we now define  $\rho'(C_1, C_2, \delta, J)$  to be twice the infimum of  $R'_P(C_2)$  over all polynomials  $P$  such that  $|P(x) - J(x)| \leq \delta$  for every  $x \in [-C_1, C_1]$ .

**Lemma 4.7.** *Let  $\|\cdot\|$  be a QAP-norm, let  $J : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function and let  $C_1, C_2$  and  $\delta$  be positive real numbers, with  $C_1 = C(1)C_2$ . Then there exists a polynomial  $P$  such that  $\|P\phi - J\phi\|_\infty \leq \delta$  and  $\|P\phi\|^* \leq \rho'(C_1, C_2, \delta, J)$  for every  $\phi \in \mathbb{R}^n$  such that  $\|\phi\|_{BAC}^* \leq C_2$ .*

*Proof.* It is immediate from the definition of  $\rho'(C_1, C_2, \delta, J)$  that for every  $C_1, C_2$  and  $\delta$  there exists a polynomial  $P$  such that  $|P(x) - J(x)| \leq \delta$  for every  $x \in [-C_1, C_1]$ , and such that  $R'_P(C_2) \leq \rho'(C_1, C_2, \delta, J)$ .

Next, observe that if  $X$  is the set specified in the definition of QAP-norms, and  $f$  is a function in  $X$ , then  $\|\mathcal{D}f\|^* \leq C(1)$ , by property (iii). Therefore, if  $\phi \in \mathbb{R}^n$  is a function with  $\|\phi\|_{BAC}^* \leq C_2$ , it follows that  $\|\phi\|^* \leq C(1)C_2 = C_1$ . Then  $\|\phi\|_\infty \leq C_1$  as well, from the definition of QAP-norms. Since  $P$  and  $J$  agree to within  $\delta$  on  $[-C_1, C_1]$ , it follows that  $\|P\phi - J\phi\|_\infty \leq \delta$ .

From the formula for  $\|\phi\|_{BAC}^*$  and the fact that this is at most  $C_2$  it follows that for any  $\epsilon > 0$  we can write  $\phi$  as a linear combination of basic anti-uniform functions, with the absolute values of the coefficients adding up to at most  $C_2 + \epsilon$ . Therefore, for any  $\epsilon > 0$  we can write  $\phi^m$  as a linear combination of products of  $m$  basic anti-uniform functions, with the absolute values of the coefficients adding up to at most  $C_2^m + \epsilon$ . Each of these products has  $\|\cdot\|^*$ -norm at most  $C(m)$ , by property (iii) of QAP-norms. Hence, by the triangle inequality,  $\|\phi^m\|^* \leq C(m)C_2^m$ . More generally, if  $P$  is the polynomial  $P(x) = a_nx^n + \cdots + a_1x + a_0$ , then by the triangle inequality we obtain that

$$\begin{aligned} \|P\phi\|^* &\leq |a_n|\|\phi^n\|^* + \cdots + |a_1|\|\phi\|^* + |a_0| \\ &\leq C(n)|a_n|C_2^n + \cdots + C(1)|a_1|C_2 + |a_0| \\ &= R'_P(C_2). \end{aligned}$$

As we remarked at the beginning of the proof, this is at most  $\rho'(C_1, C_2, \delta, J)$ , so the lemma is proved.  $\square$

**Theorem 4.8.** *Let  $\mu$  and  $\nu$  be non-negative functions on  $\{1, 2, \dots, n\}$  such that  $\|\mu\|_1$  and  $\|\nu\|_1$  are both at most 1, and let  $\eta, \delta > 0$ . Let  $\|\cdot\|$  be a QAP-norm on  $\mathbb{R}^n$ , with respect to the set  $X$  of all functions  $f \in \mathbb{R}^n$  such that  $|f(x)| \leq \max\{\mu(x), \nu(x)\}$  for every  $x$ . Let  $J : \mathbb{R} \rightarrow \mathbb{R}$  be the function given by  $J(x) = (x + |x|)/2$  and let  $\epsilon = \delta/2\rho'(C(1)c(\eta)^{-1}, c(\eta)^{-1}, \delta/4, J)$ , where  $\rho'$  is defined as in the discussion just above. Suppose that  $\|\mu - \nu\| \leq \epsilon$ . Then for*

every function  $f$  with  $0 \leq f \leq \mu$  there exists a function  $g$  such that  $0 \leq g \leq \nu(1 - \delta)^{-1}$  and  $\|f - g\| \leq \eta$ .

*Proof.* An equivalent way of stating the conclusion is that  $f = g + h$  with  $0 \leq g \leq \nu(1 - \delta)^{-1}$  and  $\|h\| \leq \eta$ . Since such an  $h$  will belong to  $X$ , we know that a sufficient condition for  $\|h\|$  to be at most  $\eta$  is that  $\|h\|_{BAC}$  is at most  $c(\eta)$  (since  $c$  is strictly decreasing). Thus, if the result is false, then we can find a functional  $\phi$  such that  $\langle f, \phi \rangle > 1$ , but  $\langle g, \phi \rangle \leq 1$  for every  $g$  such that  $0 \leq g \leq \nu(1 - \delta)^{-1}$ , and  $\|\phi\|_{BAC}^* \leq c(\eta)^{-1}$ . For the rest of the proof, we shall write  $C_2$  for  $c(\eta)^{-1}$ .

As in the proof of Theorem 4.3, the first condition on  $\phi$  is equivalent to the statement that  $\langle \nu, \phi_+ \rangle \leq 1 - \delta$ , and we still have that  $\phi_+ = J\phi$ . Also, Lemma 4.7 gives us a polynomial  $P$  such that  $\|P\phi - J\phi\|_\infty \leq \delta/4$  and  $\|P\phi\|^* \leq \rho' = \rho'(C(1)C_2, C_2, \delta/4, J)$ .

Since  $\langle \nu, \phi_+ \rangle \leq 1 - \delta$  and  $\|\nu\|_1 \leq 1$ , it follows that  $\langle \nu, P\phi \rangle \leq 1 - 3\delta/4$ . Since  $\|P\phi\|^* \leq \rho'$  and  $\|\mu - \nu\| \leq \epsilon$ , it follows that  $\langle \mu, P\phi \rangle \leq 1 - 3\delta/4 + \epsilon\rho'$ . Since  $\|\mu\|_1 \leq 1$ , it follows that  $\langle \mu, \phi_+ \rangle \leq 1 - \delta/2 + \epsilon\rho'$ . Since  $f \leq \mu$  it follows that  $\langle f, \phi_+ \rangle \leq 1 - \delta/2 + \epsilon\rho'$ , and since  $f \geq 0$  it follows that  $\langle f, \phi \rangle \leq 1 - \delta/2 + \epsilon\rho'$ , which is a contradiction.  $\square$

The abstract theorem stated and proved by Tao and Ziegler is both more and less general than Theorem 4.8. It is less general in that it takes  $\nu$  to be the uniform probability measure (and uses the letter  $\nu$  instead of  $\mu$ , so that the two measures are  $\nu$  and  $\mathbf{1}$ ). But in a small way it is more general: they observe that we did not really need the full strength of the assumptions we made.

#### 4.4. Arithmetic progressions in the primes.

In this section we shall briefly describe how a special case of Theorem 4.8, the second transference principle we proved earlier in the paper, was used by Green and Tao to prove that the primes contain arbitrarily long arithmetic progressions.

The main idea of their proof is an ingenious way of getting round the difficulty that the primes less than  $N$  do not form a dense subset of  $\{1, 2, \dots, N\}$ . This sparseness problem occurs in several places in the literature, and there is a method by which one can sometimes deal with it, which is to exploit the fact that one has a lot of control over *random* (or random-like) sets. In particular, there are various results that assert that if  $X$  is a sparse random-like set and  $Y$  is a subset of  $X$  that is dense in  $X$  (in the sense that  $|Y|/|X|$  is bounded below by a positive constant) then  $Y$  behaves in a way that is

analogous to how a dense set would behave. That is, sparse sets can be handled if you can embed them densely into random-like sets.

Green and Tao reasoned that an approach like this might work for the primes. There is a standard technicality to deal with first, which is that the primes are much denser in some arithmetic progressions than others. A moment's thought shows that this makes it impossible to embed the primes from 1 to  $N$  densely into a quasirandom set. However, one can restrict to an arithmetic progression in which the primes are particularly dense (by looking at primes that are congruent to  $a \pmod{m}$ , where  $m$  is the product of the first few primes and  $a$  is coprime to  $m$ ), in which this problem effectively disappears.

To carry out their approach, they needed to do two things. First, they had to prove that there was indeed a quasirandom set containing the primes (inside a suitable arithmetic progression, but we'll use the word "primes" as a shorthand here) that was not much bigger than the primes. If they could do that, then the general principle that relatively dense subsets of quasirandom sets behave like dense sets would suggest that the primes should behave like a dense set. Since dense sets contain plenty of arithmetic progressions, so should the primes. The second stage of their proof was to make this heuristic argument rigorous.

As it turns out, they did not construct a quasirandom superset of the primes, but an object that they called a *pseudorandom measure*. This was a non-negative function  $\nu$  that did not have to be 01-valued, but in other respects behaved like a superset of the primes. (In fact, they normalized it to have average 1, but even then it did not take just one non-zero value.) The construction of  $\nu$  was based on very recent (at the time) results of Goldston and Yıldırım [GY]. This part of the proof belongs squarely in analytic number theory and we shall say no more about it here.

The other part of the proof proceeded as follows. Let  $\nu$  be a pseudorandom measure: that is, a non-negative function defined on  $\{1, 2, \dots, N\}$  such that  $\|\nu\|_1 = 1$ , which satisfied certain quasirandom properties. (These properties were similar to, but stronger than, the assertion that  $\|\nu - \mathbf{1}\|_{U^k}$  was very small.) Let us call a set  $A$  *dense relative to  $\nu$*  if there is a positive constant  $\lambda$  such that  $\lambda A \leq \nu$  and  $\|\lambda A\|_1 \geq c$  for some positive constant  $c$  that does not depend on  $N$ . Since  $\|\nu - \mathbf{1}\|_{U^k}$  is small, the transference principle of Theorem 4.8 can be used to replace the function  $\lambda A$  by a function  $f$  that takes values in  $[0, 1]$  and has the property that  $\|f - \lambda A\|_{U^k}$  is small, provided, that is, that the hypotheses of Theorem 4.8 are satisfied.

The programme for completing the proof is therefore clear: one must prove that the hypotheses are indeed satisfied, and one must prove that the fact that  $\|f - \lambda A\|_{U^k}$  is small allows us to conclude that  $A$  contains arithmetic progressions of length  $k + 2$  (as one expects, since in other contexts the  $U^k$  norm controls progressions of this length).

Let us briefly recall what these hypotheses are. We define  $X$  to be the set of all functions that are bounded above in modulus by  $\nu + 1$ , and we would like the  $U^k$  norm to be a QAP-norm with respect to  $X$ . (These were defined at the beginning of Section 4.3.) Not surprisingly, as our non-linear operator  $\mathcal{D}$  we take the operator defined just before Proposition 4.5 (for the appropriate  $k$ ), except that for convenience we multiply it by  $2^{-(k+1)}$ .

The first hypothesis is that  $\langle f, \mathcal{D}f \rangle \leq 1$  for every  $f \in X$ . It is straightforward to check from Green and Tao's definition of pseudorandomness that  $\|f\|_{U^k}$  is at most  $2^k + o(1)$  for each  $f \in X$ , and therefore this hypothesis is satisfied.

The second is that  $\langle f, \mathcal{D}f \rangle \geq c(\epsilon)$  for every  $f \in X$  with  $\|f\|_{U^k} \geq \epsilon$ . But this is true because, with our definition of  $\mathcal{D}$ ,  $\langle f, \mathcal{D}f \rangle = 2^{-(k+1)}\|f\|_{U^k}$ . (We have essentially given this argument already, in Proposition 4.5.)

The third is that products of basic anti-uniform functions have bounded  $(U^k)^*$ -norms. This is a lemma of Green and Tao that we stated as Lemma 4.6. It should be noted that to prove this they required quasirandomness hypotheses on  $\nu$  that are stronger than one might expect: in particular they needed more than just that  $\nu$  should be close to  $\mathbf{1}$  in some  $U^r$  norm. (The precise condition they needed is called the *correlation condition* in their paper.) It is not known whether there exists an  $r$  such that their transference theorem holds under the hypothesis that  $\|\nu - \mathbf{1}\|_{U^r}$  is small.

The one remaining ingredient of their argument is what they call a “generalized von Neumann theorem,” in which they establish the fact mentioned above, that if  $\|f - \lambda A\|_{U^k}$  is small then  $A$  contains arithmetic progressions of length  $k + 2$ . More precisely,

$$\lambda^{k+2} \mathbb{E}_{x,d} A(x)A(x+d)\dots A(x+(k+1)d) \approx \mathbb{E}_{x,d} f(x)f(x+d)\dots f(x+(k+1)d).$$

If  $A$  is a dense set, so that  $\lambda$  is bounded above by a constant independent of  $N$ , then this is a standard result, but it is quite a bit harder to prove when all one knows about  $A$  is that  $\lambda A$  is bounded above by a pseudorandom measure.

## 5. TAO'S STRUCTURE THEOREM.

In this section we shall combine some of the methods and results of previous sections in order to obtain a general structure theorem for bounded functions. This result resembles

Proposition 3.7 in that we decompose a function  $f$  as a sum  $f_1 + f_2 + f_3$  with  $\|f_1\|^* \leq C$ ,  $\|f_2\| \leq \eta(C)$  and  $\|f_3\|_2 \leq \epsilon$ , but this time we shall assume that  $f$  takes values in an interval  $[a, b]$  and deduce stronger properties of the functions  $f_i$ : in particular,  $f_1$  will also take values in the interval  $[a, b]$ . In order to do this, we shall need to use polynomial approximations. It would be possible to prove a result about QAP-norms, but the notation is simpler if we assume the stronger hypothesis that the dual norm  $\|\cdot\|^*$  is an algebra norm. As we shall see, this result is general enough to apply in many interesting situations.

Here, then, is the structure theorem we shall prove in this section. Tao's structure theorem is essentially the same result, but for a specific sequence of algebra norms. However, his method can easily be modified to prove this more general formulation. (In other words, the point of this section is the method of proof rather than the extra generality of the conclusion.) We should mention here that there are other results of a similar flavour to Tao's, which are often referred to as "arithmetic regularity lemmas". The following result can be thought of as an abstract arithmetic regularity lemma.

**Theorem 5.1.** *Let  $\|\cdot\|$  be a norm defined on  $\mathbb{R}^n$ , and suppose that the dual norm  $\|\cdot\|^*$  is an algebra norm. Let  $f \in \mathbb{R}^n$  be a function that takes values in the interval  $[a, b]$ . Let  $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a positive decreasing function and let  $\epsilon > 0$ . Then there is a constant  $C_0$ , depending on  $\eta$  and  $\epsilon$  only, such that  $f$  can be written as a sum  $f_1 + f_2 + f_3$ , with  $\|f_1\|^* \leq C_0$ ,  $\|f_2\| \leq \eta(\|f_1\|^*)$ , and  $\|f_3\|_2 \leq \epsilon$ . Moreover,  $f_1$  and  $f_1 + f_3$  both take values in  $[a, b]$ .*

The last condition may look slightly strange, but it is important in applications. For instance, for Tao's application to Szemerédi's theorem,  $[a, b]$  is the interval  $[0, 1]$ , and  $f_1$  is the "structured part" of  $f$ . The key step in his argument is that  $\mathbb{E}_x f_1 \geq \delta$  implies that  $\mathbb{E}_{x,d} f_1(x) f_1(x+d) \dots f_1(x+(k-1)d) \geq c(\delta) > 0$ , and more generally that the same is true of  $f_1 + f_3$ : that is, after a small  $L_2$ -perturbation of the function  $f_1$ . However,  $c(\delta)$  is much smaller than  $\delta$ ; as a result, it is crucial that both  $f_1$  and  $f_1 + f_3$  should be positive, so that  $c(\delta)$  is not swamped by a negative error term.

There is a simple way of making Theorem 5.1 more general, and this is very important for some applications, including Tao's application to Szemerédi's theorem. In order to explain the generalization, it will be convenient to introduce another definition.

**Definition.** *Let  $\|\cdot\|$  and  $|\cdot|^*$  be two norms on  $\mathbb{R}^n$  and let  $c : (0, 1] \rightarrow (0, 1]$  be a strictly increasing function. Then  $|\cdot|^*$  is an approximate dual (at rate  $c$ ) for  $\|\cdot\|$  if the following two conditions hold:*

- (i)  $\langle f, \phi \rangle \leq \|f\| |\phi|^*$  for any two functions  $f$  and  $\phi$  in  $\mathbb{R}^n$ ;
- (ii) if  $\|f\|_\infty \leq 1$  and  $\|f\| \geq \epsilon$  then there exists  $\phi \in \mathbb{R}^n$  such that  $|\phi|^* \leq 1$  and  $\langle f, \phi \rangle \geq c(\epsilon)$ . (Equivalently,  $|f| \geq c(\epsilon)$ , where  $|\cdot|$  is the predual of  $|\cdot|^*$ .)

The first of these estimates is equivalent to the assertion that  $\|\phi\|^* \leq |\phi|^*$  for every  $\phi \in \mathbb{R}^n$ . The second is equivalent to the assertion that if  $\|f\|_\infty \leq 1$ , then  $|f| \geq c(\|f\|)$ . Therefore, if a norm  $\|\cdot\|$  merely has an *approximate* dual  $|\cdot|^*$  that is an algebra norm, we can apply Theorem 5.1 to the norm  $|\cdot|$  and conclude that  $|f_1|^* \leq C_0$ ,  $\|f_2\| \leq c^{-1}(\eta(|f_1|^*))$  and  $\|f_3\| \leq \epsilon$ . Since  $\eta$  can be chosen to tend to zero arbitrarily fast, so can  $c^{-1} \circ \eta$ . Thus, Theorem 5.1 has the following immediate corollary.

**Corollary 5.2.** *Let  $\|\cdot\|$  and  $|\cdot|$  be two norms on  $\mathbb{R}^n$  and suppose that  $|\cdot|^*$  is an approximate dual for  $\|\cdot\|$ . Let  $f$  be a function that takes values in an interval  $[a, b]$ . Let  $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a positive decreasing function and let  $\epsilon > 0$ . Then there is a constant  $C_0$ , depending only on  $\eta$ ,  $\epsilon$  and the function  $c$  that appears in the specification of the approximate duality, such that  $f$  can be written as a sum  $f_1 + f_2 + f_3$ , with  $|f_1|^* \leq C_0$ ,  $\|f_2\| \leq \eta(\|f_1\|^*)$ , and  $\|f_3\|_2 \leq \epsilon$ . Moreover,  $f_1$  and  $f_1 + f_3$  both take values in  $[a, b]$ .*

### 5.1. A proof of the structure theorem.

The proof we shall give in this paper is quite different from that of Tao. The main idea is to start with a decomposition obtained using Proposition 3.7 (which was an easy consequence of the Hahn-Banach theorem) and to adjust it until the functions  $f_1$  and  $f_1 + f_3$  have the right ranges. During the process of adjustment, we shall have cause to use Theorem 4.3, the first of the transference theorems obtained in the previous section. The proof is conceptually very simple, but it involves a longish sequence of small calculations to check that the errors that we introduce when we adjust our decomposition are small.

To begin with, then, let  $\theta$  be a decreasing positive function and  $\beta$  a positive constant, both to be specified later, and apply Proposition 3.7 to write  $f$  as  $f_1 + f_2 + f_3$  with  $\|f_1\|^* = K$ ,  $\|f_2\| \leq \theta(K)$  and  $\|f_3\| \leq \beta$ . Here,  $K$  is bounded above by a function of  $\theta$  and  $\beta$ , so later we shall need  $\theta$  and  $\beta$  to depend only on  $\eta$  and  $\epsilon$ .

We would now like to modify  $f_1$  so that it takes values in the interval  $[a, b]$ . The obvious way of doing this is to apply Lemma 4.2 with the continuous function  $J$  that takes the value  $a$  when  $x < a$ ,  $b$  when  $x > b$  and  $x$  when  $x \in [a, b]$ . This gives us a new function  $Pf_1$  such that  $\|Pf_1 - Jf_1\|_\infty \leq \delta$  and  $\|Pf_1\|^* \leq \rho = \rho(K, \delta, J)$ . The first inequality implies that  $Pf_1$  takes values in  $[a - \delta, b + \delta]$ , and a small adjustment will correct that to  $[a, b]$ .

However, before we do the adjustment, let us check that  $f_1 - Pf_1$  is small in an appropriate sense. Intuitively, this is plausible: it should not be possible for the structured part of  $f$  to stray too far from the interval  $[a, b]$  for too long. This intuition turns out to be correct, and proving it rigorously is not very hard.

**Lemma 5.3.** *Let  $f_1$  and  $Pf_1$  be the functions just defined. Then provided that the function  $\theta$  is sufficiently small (in terms of  $a, b$  and  $\beta$ ), we have the inequality  $\|f_1 - Pf_1\|_2 \leq 3\beta/2$ .*

*Proof.* From the decomposition  $f = f_1 + f_2 + f_3$  we obtain the decomposition

$$f_1 - Pf_1 = (f - Pf_1) - f_2 - f_3$$

We shall now bound  $\|f_1 - Pf_1\|_2^2$  by looking at the inner products of  $f_1 - Pf_1$  with each of the three terms on the right-hand side.

First of all, if  $f_1(x) > b$  then  $Jf_1(x) = b$ , so  $|Pf_1(x) - b| \leq \delta$ , and therefore  $f_1(x) - Pf_1(x) \geq -\delta$ . Since  $f$  takes values in  $[a, b]$ , we also find that  $f(x) - Pf_1(x) \leq \delta$ . Similarly, if  $f_1(x) < a$  then we find that  $f_1(x) - Pf_1(x) \leq \delta$  and  $f(x) - Pf_1(x) \geq -\delta$ . If  $f_1(x) \in [a, b]$ , then  $f_1(x) = Jf_1(x)$ , so  $|f_1(x) - Pf_1(x)| \leq \delta$ , and  $|f(x) - Pf_1(x)| \leq (b - a + 2\delta)$ . It follows from these three estimates that  $\langle f_1 - Pf_1, f - Pf_1 \rangle \leq 4\delta^2 + \delta(b - a)$ .

Since  $\|f_1\|^* \leq K$ ,  $\|Pf_1\|^* \leq \rho$ , and  $\|f_2\| \leq \theta(K)$ , it follows that

$$|\langle f_1 - Pf_1, f_2 \rangle| \leq (K + \rho)\theta(K).$$

For the third inner product we use Cauchy-Schwarz to give a trivial implicit estimate:

$$|\langle f_1 - Pf_1, f_3 \rangle| \leq \beta\|f_1 - Pf_1\|_2.$$

From the estimates for these inner products it follows that

$$\|f_1 - Pf_1\|_2^2 \leq 4\delta^2 + \delta(b - a) + (K + \rho)\theta(K) + \beta\|f_1 - Pf_1\|_2.$$

Therefore, if we choose  $\delta$  such that  $4\delta^2 + \delta(b - a) \leq \beta^2/4$  and  $\theta$  in such a way that  $(K + \rho(K, \delta, J))\theta(K) \leq \beta^2/2$  for every  $K$ , then

$$\|f_1 - Pf_1\|_2^2 \leq \beta^2/4 + \beta^2/2 + \beta\|f_1 - Pf_1\|_2,$$

from which it follows, on completing the square, that  $\|f_1 - Pf_1\|_2 \leq 3\beta/2$ , as claimed. To complete the proof, note that the condition on  $\theta$  depends on  $\rho$ , and hence on  $\delta$ , and  $\delta$  depends on  $a, b$  and  $\beta$ .  $\square$

The next step is very simple. Let  $L$  be the linear function that takes  $a - \delta$  to  $a$  and  $b + \delta$  to  $b$ . Then  $|L(x) - x|$  is at most  $\delta$  for every  $x$  in the interval  $[a - \delta, b + \delta]$ . Since  $f_1$  takes

values in this interval, it follows that  $\|LPf_1 - Pf_1\|_\infty \leq \delta$ . Also, if we write  $L(x) = \lambda x + \mu$ , it is easy to see that  $0 < \lambda < 1$ , from which it follows that  $\|LPf_1\|^* \leq \|Pf_1\|^* + \mu$  (since  $|\mathbf{1}|^* = 1$ ). A small calculation shows that  $\mu = -(a+b)\delta/(a-b-2\delta)$ , so  $\|LPf_1\|^* \leq 2\rho$ , provided  $\delta$  is moderately small (depending on  $a$  and  $b$ ).

Let us now see where we have reached. We started with a decomposition  $f = f_1 + f_2 + f_3$ , and we have now modified  $f_1$ , first to  $Pf_1$  and then to  $LPf_1$ . The first modification incurred an extra error of  $L_2$ -norm at most  $3\beta/2$ , and the second an extra error of  $L_\infty$ -norm, and hence  $L_2$ -norm, at most  $\delta$ . If we assume that  $\delta \leq \beta/2$  then we find that we have a decomposition  $f = g_1 + g_2 + g_3$ , where  $g_1 = LPf_1$ ,  $g_2 = f_2$ , and  $g_3 = f_3 + (f_1 - LPf_1)$ . We have shown that  $\|g_1\|^* \leq 2\rho$ , that  $\|g_2\| = \|f_2\| \leq \theta(K)$ , and that  $\|g_3\|_2 \leq \beta + 3\beta/2 + \beta/2 = 3\beta$ . Moreover,  $g_1$  takes values in the interval  $[a, b]$ .

This gives us most of what we want (if we choose  $\beta$  and  $\theta$  appropriately). The main thing we are missing is any information about the range of  $g_1 + g_3$ . In order to obtain the extra property that  $g_1 + g_3$  takes values in  $[a, b]$ , we shall focus on the equivalent problem of ensuring that  $f(x) - b \leq g_2(x) \leq f(x) - a$  for every  $x$ .

Note that  $f(x) - b \leq 0$  and  $f(x) - a \geq 0$  for every  $x$ . Our strategy for obtaining these bounds on  $g_2$  is even simpler than our strategy for adjusting  $f_1$  earlier: we shall replace  $g_2(x)$  by  $f(x) - a$  whenever  $g_2(x) > f(x) - a$ , and similarly on the other side. However, if that is all we do then we lose all information about  $\|g_2\|$ . This is where Theorem 4.3, our first transference theorem, comes in: when we adjust the positive part of  $g_2$  we can use Theorem 4.3 to make a complementary adjustment to the negative part, and vice versa.

Let us therefore set  $g'_2(x)$  to be  $\min\{g_2(x), f(x) - a\}$  for each  $x$ . First we need a simple lemma.

**Lemma 5.4.** *If  $g'_2 = \min\{g_2, f - a\}$ , then  $\|g_2 - g'_2\|_2 \leq 3\beta$ .*

*Proof.* For every  $x$ , either  $g_2(x) - g'_2(x) = 0$  or

$$0 \leq g_2(x) - g'_2(x) = g_2(x) - f(x) + a = a - g_1(x) - g_3(x) \leq -g_3(x),$$

where the last inequality follows from the fact that  $g_1(x) \in [a, b]$  for every  $x$ . It follows that  $\|g_2 - g'_2\|_2 \leq \|g_3\|_2$ , which we have established to be at most  $3\beta$ .  $\square$

Our first attempt at adjusting the decomposition is to write

$$f = g_1 + g'_2 + (g_3 + g_2 - g'_2).$$

Our main problem now is that we do not have a good estimate for  $\|g'_2\|$ . To deal with this, we shall adjust the negative part of  $g_2$  as well, using Theorem 4.3. Let  $\mu = (g_2)_+$

and let  $\nu = (g_2)_-$ . Then  $\mu$  and  $\nu$  are disjointly supported, so both  $\|\mu\|_1$  and  $\|\nu\|_1$  are at most  $\|g_2\|_1$ , which is at most  $\|g_2\|_2$ . Since  $g_2 = (f - g_1) + g_3$  and  $f$  and  $g_1$  take values in  $[a, b]$ ,  $\|g_2\| \leq |b - a| + 3\beta$ , by the triangle inequality and our estimate for  $\|g_3\|_2$ . Let  $\alpha = |b - a| + 3\beta$ .

We now apply Theorem 4.3 with  $\mu$  and  $\nu$  as above and with  $f = (g'_2)_+$ . Strictly speaking, this is not quite accurate, since the upper bounds for  $\|\mu\|_1$  and  $\|\nu\|_1$  are  $\alpha$  rather than 1, but we can look at the functions  $\alpha^{-1}\mu$ ,  $\alpha^{-1}\nu$  and  $\alpha^{-1}f$  instead. The main hypothesis we have is that  $\|g_2\| = \|\mu - \nu\| \leq \theta(K)$ , so we can take  $\epsilon$  to be  $\alpha^{-1}\theta(K)$  in Theorem 4.3. If  $\tau > 0$  is a constant such that  $\alpha^{-1}\theta(K) = \delta/2\rho(\alpha\tau^{-1}, \beta/4, J)$  (where now  $J(x)$  is the function  $(x + |x|)/2$ ), then we may conclude that there is a function  $g$  such that  $0 \leq g \leq \nu(1 - \beta)^{-1}$  and  $\|f - g\| \leq \tau$ . The important thing to note here is that  $\tau$  tends to zero as  $\theta(K)$  tends to zero.

Define  $g''_2$  to be  $f - g$ . This gives us a decomposition

$$f = g_1 + g''_2 + [g_3 + (g_2 - g'_2) + (g'_2 - g''_2)].$$

We have the upper bounds  $g''_2(x) \leq f(x) - a$  for every  $x$ , and  $\|g''_2\| \leq \tau$ . However, we have not yet checked that  $\|g'_2 - g''_2\|_2$  is small. For this we need another simple lemma.

**Lemma 5.5.** *Let  $\nu \in \mathbb{R}^n$  be a non-negative function and suppose that  $\nu$  can be written as a sum  $\nu_1 + \nu_2$ , where  $\|\nu_1\|_\infty \leq \alpha$  and  $\|\nu_2\|_2 \leq \gamma$ . Then  $\|h\|_2 \leq \gamma + (\alpha\|h\|_1)^{1/2}$  for any function  $h$  with  $0 \leq h \leq \nu$ .*

*Proof.* By the positivity of  $h$  and  $\nu$ ,

$$\|h\|_2^2 \leq \langle h, \nu_1 + \nu_2 \rangle \leq \alpha\|h\|_1 + \gamma\|h\|_2.$$

The bound stated is an easy consequence of this. □

**Corollary 5.6.** *Let  $g''$  be any function such that  $g''(x) = g'_2(x)$  when  $g'_2(x)$  is non-negative, and  $0 \geq g''(x) \geq g'_2(x)$  otherwise. Suppose also that  $\|g_2\| \leq \tau$ . Then  $\|g'' - g'_2\|_2 \leq 3\beta + (\alpha(\tau + 3\beta))^{1/2}$ .*

*Proof.* It follows from the hypotheses that  $0 \leq g''(x) - g'_2(x) \leq \nu(x)$  for every  $x$ . Recall that  $g_2 = (f - g_1) + g_3$  and that  $\|f - g_1\|_\infty \leq b - a$ . It follows easily that  $\nu = (g_2)_-$  satisfies the conditions of Lemma 5.5, with  $\gamma = 3\beta$ . (We could improve  $\alpha$  to  $b - a$ , but this is not worth bothering about.)

Applying the lemma, we deduce that  $\|g'' - g'_2\|_2 \leq 3\beta + (\alpha\|g'' - g'_2\|_1)^{1/2}$ . Now let us turn our attention to bounding  $\|g'' - g'_2\|_1$ . Since  $\|f - g\| \leq \tau$  and  $\|\cdot\|^*$  is an algebra

norm, it follows from Lemma 4.1 that  $|\mathbb{E}_x(f(x) - g(x))| \leq \tau$ . But  $|\mathbb{E}_x(f(x) - g(x))| \geq \|g'' - g'_2\|_1 - \|g_2 - g'_2\|_1$ , since  $g'' \geq g'_2$ , and  $\|g_2 - g'_2\|_1 \leq \|g_2 - g'_2\|_2$ , which we have already shown is at most  $3\beta$ . Therefore,  $\|g'' - g'_2\|_1 \leq \tau + 3\beta$ . Inserting this bound into the estimate at the beginning of this paragraph, we find that  $\|g'' - g'_2\|_2 \leq 3\beta + (\alpha(\tau + 3\beta))^{1/2}$ , as claimed.  $\square$

Since the function  $g''_2$  constructed earlier satisfies the hypotheses required of  $g''$  in Corollary 5.6, we now have an improved decomposition  $f = h_1 + h_2 + h_3$ , where  $h_1 = g_1$ ,  $h_2 = g''_2$  and  $h_3 = g_3 + g_2 - g''_2$ . Our arguments so far have shown that  $h_1$  takes values in  $[a, b]$ , that  $\|h_1\|^* \leq 2\rho$ , that  $\|h_2\| \leq \tau$ , that  $h_2(x) \leq f(x) - a$  for every  $x$ , and that  $\|h_3\|_2 \leq 6\beta + 3\beta + (\alpha(\tau + 3\beta))^{1/2}$ . If  $\beta$  is sufficiently small (depending on  $b - a$  if that is small, which in a typical application it will not be), and  $\tau$  is sufficiently small (depending on  $\beta$ ), then this last quantity is at most  $2(\beta(b - a))^{1/2}$ , which we shall call  $\zeta$ .

From the way we constructed  $h_2$ , we know that the sign of  $h_2$  is the same as that of  $g_2$ , and that  $|h_2(x)| \leq |g_2(x)|$  for every  $x$ . We have obtained the upper bound of  $f - a$  that we wanted for  $h_2$ ; now we need a further adjustment in order to obtain a lower bound of  $f - b$ . It is obvious how to do this: we shall sketch the argument only very briefly.

First, we let  $h'_2(x) = \max\{h_2(x), f(x) - b\}$  for every  $x$ . Then a simple modification of Lemma 5.4 shows that  $\|h_2 - h'_2\|_2 \leq 3\zeta$ .

Next, we use Theorem 4.3 to reduce the positive part of  $h'_2$ , while leaving the negative part unchanged, to create a function  $h''_2$  with  $\|h''_2\|$  small. If we let  $\alpha' = b - a + 3\zeta$ , then the same argument as before gives us an upper bound  $\|h''_2\| \leq \kappa$ , where  $\kappa$  is a constant such that  $\alpha'^{-1}\tau = \delta/2\rho(\alpha'\kappa^{-1}, \zeta/4, J)$ . In particular,  $\kappa$  tends to zero as  $\tau$  tends to zero.

Next, a simple modification of Corollary 5.6 tells us that  $\|h''_2 - h'_2\|_2 \leq 3\zeta + (\alpha'(\kappa + 3\zeta))^{1/2}$ . Therefore, we have a decomposition  $f = u_1 + u_2 + u_3$ , with  $u_1 = h_1$ ,  $u_2 = h''_2$  and  $u_3 = h_3 + h_2 - h''_2$ . Since  $u_1 = h_1$ , it takes values in  $[a, b]$ . The construction of  $h''_2$  guarantees that  $f(x) - b \leq u_2(x) \leq f(x) - a$ , and hence that  $u_1 + u_3$  takes values in  $[a, b]$ . Finally, we have the estimates  $\|u_1\|^* \leq 2\rho$ ,  $\|u_2\| \leq \kappa$ , and  $\|u_3\|_2 \leq 6\zeta + 3\zeta + (\alpha'(\kappa + 3\zeta))^{1/2}$ . If  $\zeta$  is small enough (depending on  $b - a$ ) and  $\kappa$  is small enough (depending on  $\zeta$ ), then this last quantity is at most  $2((b - a)\zeta)^{1/2}$ .

Now let us see why these estimates are enough, recalling from the beginning of the proof that we are free to choose  $\beta$  and  $\theta$ . To begin with, we need  $2((b - a)\zeta)^{1/2}$  to be at most  $\epsilon$ . But  $\zeta$  tends to zero with  $\beta$ , so this is easily achieved. Next, recall that  $\rho = \rho(K, \delta, J)$ . We would like  $\kappa$  to be at most  $\eta(\rho)$ , which we shall ensure by making a suitable choice of  $\theta$ . The constant  $\delta$  depends on  $\beta$ ,  $a$  and  $b$  only, while  $\kappa$  tends to zero with  $\tau$ , which tends to

zero with  $\theta(K)$ . Thus, for each  $K$  we can choose  $\theta(K)$  in a way that depends on  $K, \beta, a$  and  $b$  only, such that  $\kappa \leq \eta(\rho)$ . The proof is complete.

## 5.2. Decomposition theorems with bounds on ranges.

As a simple application of Theorem 5.1, we shall now obtain the improvement that we promised earlier to our results about deducing decomposition theorems from inverse theorems. So far, we have shown that a function can be decomposed into a multiple of a convex combination of structured functions, plus an error, provided that we have a suitable inverse theorem concerning the structured functions and the kind of error we are prepared to allow. As we commented, it is sometimes useful to obtain a decomposition for which the “structured part” is bounded. We shall see that Theorem 5.1 implies rather easily that such a decomposition exists, and it has the added advantage of yielding an  $L_2$  error term rather than the  $L_1$  error term that appears in Theorem 3.8 or weaker theorems of a similar type that were discussed earlier in Section 3.2. However, the bound on the sum of the coefficients of the structured functions is very bad. For some applications, this is not a concern, but for others it turns out to be preferable to use weaker theorems.

**Theorem 5.7.** *Let  $\|\cdot\|$  be a norm on  $\mathbb{R}^n$  and let  $\Phi \subset \mathbb{R}^n$  be a set of functions satisfying the following properties for some strictly increasing function  $c : (0, 1] \rightarrow (0, 1]$ :*

- (i)  *$\Phi$  contains the constant function  $1$ ,  $\Phi = -\Phi$ ,  $\|\phi\|_\infty \leq 1$  for every  $\phi \in \Phi$ , and the linear span of  $\Phi$  is  $\mathbb{R}^n$ ;*
- (ii)  *$\langle f, \phi \rangle \leq 1$  for every  $f$  with  $\|f\| \leq 1$  and every  $\phi \in \Phi$ ;*
- (iii) *if  $\|f\|_\infty \leq 1$  and  $\|f\| \geq \epsilon$  then there exists  $\phi \in \Phi$  such that  $\langle f, \phi \rangle \geq c(\epsilon)$ .*

*Let  $\epsilon > 0$  and let  $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a strictly decreasing function. Then there is a constant  $M_0$ , depending only on  $\epsilon$  and the functions  $c$  and  $\eta$ , such that every function  $f \in \mathbb{R}^n$  that takes values in  $[0, 1]$  can be decomposed as a sum  $f_1 + f_2 + f_3$ , with the following properties:  $f_1$  and  $f_1 + f_3$  take values in  $[0, 1]$ ;  $f_1$  is of the form  $\sum_i \lambda_i \psi_i$ , where  $\sum_i |\lambda_i| = M \leq M_0$  and each  $\psi_i$  is a product of functions in  $\Phi$ ;  $\|f_2\| \leq \eta(M)$ ;  $\|f_3\|_2 \leq \epsilon$ .*

*Proof.* Let  $\Psi$  be the set of all products of functions in  $\Phi$ . Define a norm  $|\cdot|^*$  by taking  $|g|^*$  to be the infimum of all sums  $\sum_i |\lambda_i|$  such that  $g$  can be written as  $\sum_i \lambda_i \psi_i$  with every  $\psi_i$  in  $\Psi$ . It is straightforward to check that this is an algebra norm. (The fact that it is a norm rather than a seminorm relies on the boundedness of functions in  $\Phi$ , which one could in fact deduce from (ii) rather than stating as a separate assumption.) Moreover, (ii) and

(iii) imply easily that  $|\cdot|^*$  is an approximate dual for  $\|\cdot\|$ . Therefore, Corollary 5.2 implies the result.  $\square$

The following simple trick is important in applications. Property (iii) in the statement of the theorem is the assertion that there is an inverse theorem relating the norm  $\|\cdot\|$  to the set of functions in  $\Phi$ . However, the conclusion of the theorem concerns *products* of functions in  $\Phi$ , and in practice it often happens that the set  $\Phi$  of functions that one obtains from an inverse theorem is not closed under pointwise multiplication. However, it also often happens that one can give an explicit description of products of functions in  $\Phi$ , and that this description becomes only gradually less useful as the number of functions in the product increases. Under such circumstances, one can replace  $\Phi$  by the set  $\{\mathbf{1}, -\mathbf{1}\} \cup (\Phi/2)$ . This modified set clearly satisfies all the hypotheses that  $\Phi$  was required to satisfy, but now the corresponding set  $\Psi$  comes with a “penalty” of  $2^{-k}$  attached to a product of  $K$  functions. This means that the sum of the  $|\lambda_i|$  over products of significantly more than  $\log_2 M_0$  functions in  $\Phi$  make a very small (in  $L_\infty$ ) contribution to  $f_1$  and can be absorbed into the error term.

### 5.3. Applying Tao’s structure theorem

We shall not actually give applications of the structure theorem here, but merely comment on how it is applied. The rough idea, as we have already seen, is to express a bounded function (such as, for instance, the characteristic function of a dense subset of  $\mathbb{Z}_N$ ) as a sum of a structured part, a quasirandom part, and an  $L_2$  error. To do this, we need to choose a norm  $\|\cdot\|$  that measures quasirandomness in a useful way, such that its dual norm  $\|\cdot\|^*$  is an algebra norm with the property that if  $\|\phi\|^*$  is bounded then we “understand”  $\phi$  and can regard it as structured. As we have seen, a simple (but useful) example of such a norm is  $\|f\| = \|\hat{f}\|_\infty$ .

Let us briefly consider this example. If we have written a function  $f$  as  $f_1 + f_2 + f_3$  in such a way that  $\|\hat{f}\|_1 \leq C$ ,  $\|\hat{f}\|_2 \leq \eta(C)$  and  $\|f_3\|_2 \leq \epsilon$ , then we can analyse it as follows.

We first show that  $f_1$  is “approximately smooth” in the following sense. Let  $\delta, \theta > 0$  be small constants to be chosen later, and let  $K$  be the set of all  $r$  such that  $|\hat{f}_1(r)| \geq \delta$ . Since  $\|\hat{f}_1\|_2^2 = \|f_1\|_2^2 \leq 1$ , it follows that  $|K| \leq \delta^{-2}$ . Now let  $B$  be the set of all  $x \in \mathbb{Z}_N$  such that  $|\omega^{rx} - 1| \leq \theta$  for every  $r \in K$ . Sets like  $B$  are called *Bohr neighbourhoods* and have many good properties, but for now we remark merely that a fairly straightforward argument shows that the cardinality of  $B$  is at least  $\theta^{|K|} N$ .

Now let  $\beta$  be the *characteristic measure* of  $B$ : that is, the function that takes the value  $N/|B|$  on  $B$  and 0 elsewhere. This multiple of  $B$  is chosen so that  $\|\beta\|_1 = 1$ . A useful property of  $\beta$  is that  $f_1$  is close to  $f_1 * \beta$  in  $L_2$ . This can be shown with the help of Fourier transforms: the general method is known as *Bogolyubov's method*, and it is a very useful tool in additive combinatorics. We begin by observing that

$$\begin{aligned} \|f_1 - f_1 * \beta\|_2^2 &= \|\hat{f}_1 - \hat{f}_1 \hat{\beta}\|_2^2 \\ &= \sum_{r \in K} |\hat{f}_1(r)|^2 |1 - \hat{\beta}(r)|^2 + \sum_{r \notin K} |\hat{f}_1(r)|^2 |1 - \hat{\beta}(r)|^2 \end{aligned}$$

For every  $r \in K$  we have  $|1 - \hat{\beta}(r)| = |\mathbb{E}_{x \in B}(1 - \omega^{rx})| \leq \theta$ , so the first sum is at most  $\theta^2 \|\hat{f}\|_2^2 \leq \theta^2$ . We also have the trivial estimate that  $|1 - \hat{\beta}(r)| \leq 2$ , so the second sum is at most  $4\delta \|\hat{f}\|_1 \leq 4\delta C$ . Thus, by choosing  $\delta$  and  $\theta$  appropriately, we can ensure that  $f_1$  and  $f_1 * \beta$  are close in  $L_2$ , as claimed.

This tells us that for a typical pair  $x$  and  $y$ , if  $x - y \in B$ , then  $f_1(x)$  and  $f_1(y)$  are close. Equivalently,  $f_1$  is almost always roughly constant on translates of  $B$ .

Now if we choose  $\eta(C)$  to be small enough, then  $f_2$  is highly quasirandom even compared with the size of  $B$ . That is,  $\|\hat{f}_2\|_\infty$  is so small that even the restrictions of  $f_2$  to translates of  $B$  behave quasirandomly (in a sense that one can make precise in several natural ways). This means that even though  $f_2$  may have a large  $L_2$  norm, we may nevertheless think of  $f_1 + f_2$  as a tiny perturbation of  $f_1$ . For instance, if  $x$  is a typical element of  $\mathbb{Z}_N$  and  $f_1(x) \geq c$ , then the smoothness of  $f_1$  guarantees that  $f_1(y) \geq c/2$  for almost every  $y \in x + B$ . From this and the positivity of  $f_1$  it follows (if  $B$  satisfies a certain technical condition that one can always ensure) that

$$\mathbb{E}_{x,d} f_1(x) f_1(x+d) f_1(x+2d)$$

is bounded below by some (very small) positive constant related to the density of  $B$ , which depended on  $C$  only. If  $\eta(C)$  is much smaller than this constant, then perturbing by  $f_2$  cannot change this lower bound to zero.

This is not quite a sketch proof of Roth's theorem (though it is close), because there remains the problem of dealing with  $f_3$ . In fact, the correct order to work in is to think about  $f_1$  first, then  $f_1 + f_3$ , and finally  $f_1 + f_2 + f_3$ . This is why it is so helpful for  $f_1$  and  $f_1 + f_3$  to be non-negative functions.

The above idea can be thought of as a discrete analogue of at least one ergodic-theoretic proof of Roth's theorem. Tao applied his structure theorem to a sequence of cleverly

constructed algebra norms in order to extend the argument to a proof of the general case of Szemerédi's theorem. Unfortunately, the analysis of the structured function  $f_1$  becomes far harder: that is where the real difficulty of his argument lies.

The structure theorem can also be used to replace arguments that use Szemerédi's regularity lemma. This is not too surprising, as in both cases the strength of the result comes from the fact that the bound on the quasirandomness can be made so small that it is even small compared with the “natural scale” of the structured part. Similarly, it can be used to replace a version of Szemerédi's regularity lemma, due to Green [G], that concerns dense subsets of finite Abelian groups.

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